particular interest since they provide the calculation equations which may be employed with curved boundaries, while still using uniform increments in \( \Delta x \) and \( \Delta y \).

**Example 3-3**

Consider the square of Fig. 3-6. The left face is maintained at 100°C and the top face at 500°C, while the other two faces are exposed to an environment at 100°C:

\[
h = 10 \text{ W/m}^2 \cdot \text{°C} \quad \text{and} \quad k = 10 \text{ W/m} \cdot \text{°C}
\]

The block is 1 m square. Compute the temperatures of the various nodes as indicated in Fig. 3-9 and the heat flows at the boundaries.

**Solution**

The nodal equation for nodes 1, 2, 4, and 5 is

\[
T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1} - 4T_{m,n} = 0
\]

The equation for nodes 3, 6, 7, and 8 is given by Eq. (3-25), and the equation for node 9 is given by Eq. (3-26):

\[
\frac{h \Delta x}{k} = \frac{(10)(1)}{(3)(10)} = \frac{1}{3}
\]

The equations for nodes 3 and 6 are thus written

\[
2T_2 + T_5 + 567 - 4.67T_3 = 0
\]
\[
2T_3 + T_6 + 67 - 4.67T_9 = 0
\]

The equations for nodes 7 and 8 are given by

\[
2T_4 + T_8 + 167 - 4.67T_7 = 0
\]
\[
2T_5 + T_7 + T_9 + 67 - 4.67T_8 = 0
\]

and the equation for node 9 is

\[
T_6 + T_9 + 67 - 2.67T_9 = 0
\]

We thus have nine equations and nine unknown nodal temperatures. We shall discuss solution techniques shortly, but for now we just list the answers:
The heat flows at the boundaries are computed in two ways: as conduction flows for the 100 and 500°C faces and as convection flows for the other two faces. For the 500°C face, the heat flow into the face is

\[ q = \sum k \Delta x \frac{\Delta T}{\Delta y} = (10)[500 - 280.67 + 500 - 330.30 + (500 - 309.38)(\frac{1}{4})] \]
\[ = 4843.4 \text{ W/m} \]

The heat flow out of the 100°C face is

\[ q = \sum k \Delta y \frac{\Delta T}{\Delta x} = (10)[280.67 - 100 + 192.38 - 100 + (157.70 - 100)(\frac{1}{4})] \]
\[ = 3019 \text{ W/m} \]

The heat flow out the right face is given by the convection relation

\[ q = \sum h \Delta y(T - T_{\infty}) \]
\[ = (10)(\frac{1}{4})[309.38 - 100 + 217.19 - 100 + (175.62 - 100)(\frac{1}{4})] \]
\[ = 1214.6 \text{ W/m} \]

Finally, the heat flow out the bottom face is

\[ q = \sum h \Delta x(T - T_{\infty}) \]
\[ = (10)(\frac{1}{4})[(100 - 100)(\frac{1}{4}) + 157.70 - 100 + 184.71 - 100 + (175.62 - 100)(\frac{1}{4})] \]
\[ = 600.7 \text{ W/m} \]

The total heat flow out is

\[ q_{\text{out}} = 3019 + 1214.6 + 600.7 = 4834.3 \text{ W/m} \]

This compares favorably with the 4843.4 W/m conducted into the top face.

**Fig. 3-9** Nomenclature for Example 3-3.
The nodal equations may be written as

\[
\begin{align*}
T_1 & = C_1 \\
T_2 & = C_2 \\
T_3 & = C_3 \\
& \vdots \\
T_n & = C_n
\end{align*}
\] (3-27)

where \(T_1, T_2, \ldots , T_n\) are the unknown nodal temperatures. By using the matrix notation

\[
[A] = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & \vdots \\
a_{31} & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix} \quad [C] = \begin{bmatrix}
C_1 \\
C_2 \\
C_3 \\
\vdots \\
C_n
\end{bmatrix} \quad [T] = \begin{bmatrix}
T_1 \\
T_2 \\
T_3 \\
\vdots \\
T_n
\end{bmatrix}
\]

Eq. (3-27) can be expressed as

\[
[A][T] = [C] \quad (3-28)
\]

and the problem is to find the inverse of \([A]\) such that

\[
[T] = [A]^{-1}[C] \quad (3-29)
\]

Designating \([A]^{-1}\) by

\[
[A]^{-1} = \begin{bmatrix}
b_{11} & b_{12} & \cdots & b_{1n} \\
b_{21} & \cdots & \vdots & \vdots \\
b_{n1} & b_{n2} & \cdots & b_{nn}
\end{bmatrix}
\]

the final solutions for the unknown temperatures are written in expanded form as

\[
\begin{align*}
T_1 & = b_{11} C_1 + b_{12} C_2 + \cdots + b_{1n} C_n \\
T_2 & = b_{21} C_1 + \cdots \\
& \vdots \\
T_n & = b_{n1} C_1 + b_{n2} C_2 + \cdots + b_{nn} C_n
\end{align*} \quad (3-30)
\]

Clearly, the larger the number of nodes, the more complex and time-consuming the solution, even with a high-speed computer. For most conduction problems the matrix contains a large number of zero elements so that some simplification in the procedure is afforded. For example, the matrix notation for the system of Example 3-3 would be

\[
\begin{bmatrix}
-4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 2 & -4.67 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 2 & -4.67 & 0 & 0 & 1 \\
0 & 0 & 0 & 2 & 0 & 0 & -4.67 & 1 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 1 & -4.67 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -2.67
\end{bmatrix} \begin{bmatrix}
T_1 \\
T_2 \\
T_3 \\
T_4 \\
T_5 \\
T_6 \\
T_7 \\
T_8 \\
T_9
\end{bmatrix} = \begin{bmatrix}
-600 \\
-500 \\
-567 \\
-100 \\
0 \\
-67 \\
-167 \\
-67 \\
-67
\end{bmatrix}
\]
We see that because of the structure of the equations the coefficient matrix is very sparse. For this reason iterative methods of solution may be very efficient. The Gauss-Seidel method is one which we shall discuss later. An old method suitable for hand calculations with a small number of nodes is called the relaxation method. In this technique the nodal equation is set equal to some residual $\bar{q}_{m,n}$ and the following calculation procedure followed:

1. Values of the nodal temperatures are assumed.
2. The value of the residual for each node is calculated from the respective equation and the assumed temperatures.
3. The residuals are "relaxed" to zero by changing the assumptions of the nodal temperatures. The largest residuals are usually relaxed first.
4. As each nodal temperature is changed a new residual must be calculated for connecting nodes.
5. The procedure is continued until the residuals are sufficiently close to zero.

In Tables 3-3 and 3-4 relaxation solutions for the two previous examples are shown. For the most part, the relaxation method would be employed as an expedient vehicle only when a computer was not readily available.

Other methods of solution include a transient analysis carried through to steady state (see Chap. 4), direct elimination (Gauss elimination [9]), or more sophisticated iterative techniques [14]. A number of large computer programs are available for the solution of heat-transfer problems. Kern and Kraus [19] present both steady-state and transient programs which can handle up to 300 nodes. A general circuit-analysis program applicable to heat-transfer problems is available in Ref. 17, and most large computer centers have some kind of in-house program available for heat-transfer computations. Further information on numerical techniques is given in Refs. 11 to 19.

**Table 3-3** Relaxation table for system of Fig. 3-6

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<th>$\bar{q}_1$</th>
<th>$T_2$</th>
<th>$\bar{q}_2$</th>
<th>$T_3$</th>
<th>$\bar{q}_3$</th>
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