Harmonic Oscillator Project  
(Adapted from old notes)

The focus for this project is an evaluation of the generation of sinusoidal oscillations using (nearly) linear feedback circuits. The underlying principle involved is a ‘bootstrap’ process that may be described crudely as follows. The output of a unity-gain amplifier is by definition a copy of the input signal waveform that produces the output. If this output is fed back as a replacement for the input the amplifier presumably drives itself! The input necessary to start the process initially, i.e. to provide the initial amplifier output, is provided by inevitable thermal noise. Regenerative feedback maintains the oscillation.

These are however matters to consider more fully.

Discussion
There are three essential constituents of a practical harmonic oscillator:

a) An amplifier to sustain the oscillation, transferring energy (usually from a DC source) into the oscillation to replace inevitable circuit losses and oscillation energy supplied to a load;

b) Circuit components that establish the frequency of the oscillation;

c) A mechanism to define the amplitude of oscillation.

If the oscillator circuit used were truly linear there would be no amplitude limiting mechanism other than that associated with the average energy initially stored in the system. In a linear system signal amplitude can be scaled (i.e. multiplied by a constant) without affecting frequency, and the amplitude will be whatever the available energy will sustain. In a practical oscillator however there is necessarily a continual infusion of energy into the oscillation by the amplifier, and signal amplitude grows until a circuit non-linearity of one sort or another constrains further growth. Sometimes a controlled non-linearity is introduced deliberately for this purpose, rather than depending on the generally unpredictable effect of an inherent non-linearity such as amplifier saturation to limit oscillation amplitude.

Negative-Resistance Oscillator
An example of a negative resistance circuit is shown in the figure to the right. Assume an idealized opamp for simplicity, and because of this neglect the voltage difference across the amplifier input terminals, so that the voltage drops across $R_1$ and $R_2$ then are equal. With negligible amplifier input-terminal currents the circuit input volt-ampere relation is as shown in the figure. The description of the input resistance of this circuit as a ‘negative’ resistance reflects comparison of the volt-ampere relation with Ohm’s Law. However note that the idealized opamp assumption (and, although for different reasons, for that matter Ohm’s Law) has a limited range of validity in practice, in particular the range of applicability of the negative resistance relationship is constrained by amplifier saturation limits.

The resistance $R_S$ in the series RLC circuit to the right represents circuit losses, primarily the inductor wire resistance. This is ‘compensated’ by adding an equal negative resistance, i.e. a mechanism for adding energy to the circuit to replace circuit losses. A PSpice netlist for a representative design accompanies the figure.
The series-tuned LC circuit of this negative resistance oscillator involves two poles; the poles are both negative real for over-damped operation, and complex conjugate with negative real parts for under-damped operation. For the special case of zero net resistance the poles become an imaginary conjugate pair. Zero net resistance is obtained in effect by adding power to the circuit as rapidly as it is dissipated. Computed output data for oscillation, and for the cases of over- and under-damping is plotted in the figure following. Since PSpice computations do not include thermal noise ordinarily the circuit is energized by assuming an initial charge/voltage on the capacitor.

The computed frequency of oscillation $\omega$ may be compared to the expected (calculated) value obtained from $\omega^2LC = 1$. Note the initial amplifier saturation that limits the increase of energy stored in the circuit.

**Quadrature Oscillator**

The analysis in the preceding illustration is done in the frequency domain, i.e. in terms of transfer function poles and zeros. Another rather straightforward method of obtaining a sinusoidal oscillation is by assembling a circuit whose operation in the time domain emulates a second-order differential equation with constant coefficients; the natural solution of such an equation is sinusoidal. Actually it is more practical to imagine the second-order differential equation integrated twice to obtain a technically more convenient second-order integral equation; assembling electronic integrators generally is easier than assembling differentiators. The circuit diagram to the right illustrates one such an oscillator configuration.
The first stage to the left is an inverting Miller integrator. The other stage is a non-inverting integrator. The opamp (assumed operating normally) maintains the non-inverting input voltage at \( vo/2 \) (where \( vo \) is the second stage output voltage). A node equation at the capacitor node determines the current through the capacitor as shown. The integrating property of the second stage follows directly from this.

The output of the second stage is fed back to provide the input of the first, leading to a circuit with the integral equation operating description desired. Note that because the output of one stage is the integral of the output of the other stage both sine and cosine signals are available; hence the name 'quadrature' oscillator. Design and analysis of an illustrative quadrature oscillator is left as an exercise.

**Feedback Oscillators**

A particularly productive method of studying harmonic oscillators is to view them as feedback amplifiers for which the signal fed back from the output provides the entire input signal necessary to produce the output. (This raises a question of how an oscillation that drives itself starts in the first place. Although the question is fundamental the answer is rather straightforward; inevitable thermal motion of atomic charge, i.e. electrical noise, provides an inherent startup signal. The more pertinent question to consider is how the circuit must nurture this signal appropriately to sustain oscillation. The block diagram below illustrates a basic feedback system. \( S_i \) is an input signal applied to ‘summing’ node connected to an amplifier of gain \( G_S \); the amplifier output is \( S_o \). An output sample fraction \( fS_o \) is fed back and subtracted from the input to produce the net amplifier input signal \( S_i - fS_o \). The relationship between \( S_i \) and \( S_o \) is as shown.

For previous amplification applications, we assumed that the circuit designer made \( fG_S \gg 1 \), i.e designed for negative (degenerative) feedback. For this case \( S_o/S_i >1/f \). Suppose instead that a design makes \( 1+fG_S \to 0 \), implying the singular gain \( S_o/S_i \to \infty \). Whereas for degenerative feedback the signal sample is subtracted from the input the physical basis for this regenerative (positive) feedback is the addition of the output signal sample to the input, i.e. to reinforce the input signal. An increase in input signal strength further increases the output signal, and so causes still a larger reinforcing feedback signal. The condition \( 1+fG_S \to 0 \) corresponds to a theoretical feedback signal strength sufficient by itself to produce the output level necessary to support the feedback signal; the operation becomes self-sustaining. However continuing growth is not a sustainable process; the amplifier ultimately is driven into a saturated state for which the gain becomes essentially zero, and further signal growth stops.

The Barkhausen Condition \( 1+fG_S \to 0 \) is interpreted as the condition for the onset of oscillation. This is not really a 'bootstrap' affair; as a practical matter the oscillation starts because of signals generated by random electron movement (i.e. currents) associated with thermal excitation. The circuit feedback, appropriately designed, continuously reinforces a particular frequency component to form a self-sustaining output. The oscillation energy is obtained by conversion from an energy source that is an integral part of the amplifier.

As noted before circuit non-linearity, whether inherent or specifically introduced, eventually must limit the signal amplitude. Nevertheless a linear analysis is meaningfully applicable until the signal amplitude is large enough for non-linearity to be significant, and so as stated may be applied usefully to determining the conditions for the onset of oscillation. And, provided the nonlinear limiting is not too
severe, a general continuity in nature suggests the results of the linear analysis can be 'close' in some useful sense to the actual circuit performance.

**Barkhausen Conditions**
The unity loop-gain condition for the onset of oscillations (i.e. output = input) actually involves two distinct requirements: (1) the magnitude of the net loop gain must be 1, and (2) the phase of the loop gain must be 0 (or a multiple of 360°). These two independent requirements together form the Barkhausen conditions for the onset of oscillations.

In general for a particular linear circuit to support an oscillation the roots of the circuit determinant must by definition have conjugate complex poles on the imaginary axis. Hence to make an oscillator we must start with a circuit whose determinant involves at least two poles, and specify circuit parameters so that these poles are placed properly on the imaginary axis. Unfortunately a circuit with just two poles is not sufficient. The root locus for a two-pole system including loss simply does not cross the imaginary axis whatever circuit element values used. At least one more singularity, either a pole or a zero, must be present as a minimal requirement. Several oscillators meeting the minimal condition are studied here.

It can be noted that linear system may be scaled in frequency without changing the relative amplitudes of circuit voltages or currents. Frequency is involved only as a factor in a product with either a circuit inductance or capacitance, and only the product affects the voltage and current amplitudes. Hence the condition of unity loop gain magnitude can be maintained while frequency is scaled arbitrarily; simply scale inductance and capacitance by the inverse of the factor the frequency is scaled. It follows then that it is the phase of the loop gain, and only the phase, which can determine the frequency of oscillation; the oscillation frequency must be such that there is no net phase shift around the loop. Unity (or greater) loop gain magnitude is necessary to initiate the oscillation and to replace energy losses, but this requirement is quite separate from the determination of the oscillation frequency.
(A) Tuned Circuit Oscillator
The parallel tuned circuit illustrated to the right is often used in one form or another to satisfy the phase shift condition for a sinusoidal oscillator; G represents inevitable inductor (primarily) and capacitor losses more than an intended circuit element. An expression for the admittance Y looking into the circuit is shown to the right of the tuned circuit.

\[
Y = B \left\{ 1 + Q \left( \frac{s}{\omega_0} + \frac{\omega}{s} \right) \right\}
\]

where \( \omega_0^2 LC \triangleq 1 \)
and \( Q \omega_0 L G \triangleq 1 \)

A simplified parallel tuned circuit oscillator is drawn to the left. The amplifier on the left provides an adjustable output voltage, of which a fraction \( 1/(1+RY) \) is returned to drive the amplifier; Y is the admittance of the parallel combination of \( G_0, L, \) and \( C \).

Frequency selectivity is provided by the frequency-dependent voltage divider.

A straightforward analysis provides

\[
\frac{V_b}{V_a} = \frac{G}{1 + R G_0 \left\{ 1 + Q \left( \frac{s}{\omega_0} + \frac{\omega}{s} \right) \right\}}
\]

Note that Y contributes two poles and a zero to the expression, a minimal singularity requirement for oscillation to be achievable. The Barkhausen oscillation initiation condition is that \( V_b = V_a \). Note also that the expression is complex, i.e. \( s = j\omega \). The real and imaginary parts of the two sides of the expression each separately must be equal, and so provide two requirements for oscillation. The amplifier gain requirement is determined from the real part, and the frequency of oscillation from the imaginary part.

Project: Design a tuned circuit oscillator for a nominal oscillation frequency of 10 kHz and nominal peak amplitude of 6 volts or more. Use a inductor represented by a 10K\( \Omega \) resistance in parallel with 10 mH. Show explicitly how the individual Barkhausen conditions are met for your design and verify performance expectations using PSpice. Plot the amplifier voltage output, and compare it to the voltage across the tuned circuit. How does the improvement in waveform come about?

Note that the computer does not ordinarily provide thermal noise to initiate oscillation. Instead specify an initial voltage across the capacitor to provide the start-up energy.
(B) Phase-Shift Oscillator
Another circuit capable of sustaining oscillation is the RC phase-shift circuit that involves three poles. A straightforward analysis can be made as a ‘ladder’ development, i.e. note that the current into the inverting amplifier through R is \( V_a/R \), from this calculate the voltage across the capacitor in series with this \( R \) … . Obtain the equation

\[
\frac{V_b}{V_a} = 1 + \frac{6}{2} \frac{s}{RC} + \frac{5}{(sRC)^2} + \frac{1}{(sRC)^3} = G
\]

where as before \( s \) is used to represent \( j\omega \). As usual the Barkhausen conditions are obtained by equating real and imaginary parts of the two sides of the equation.

Project: Design a phase-shift oscillator for a nominal oscillation frequency of 10 kHz and nominal peak amplitude of 6 volts or more. Use \( R = 1K\Omega \). Show explicitly how the individual Barkhausen conditions are met for your design and verify circuit performance using PSpice. Comment on the transient growth of the oscillation until steady state is reached.

Note that the computer does not ordinarily provide thermal noise to initiate oscillation. Instead specify an initial voltage across one of the capacitors to provide the start-up energy.
(C) Colpitts Oscillator

The Colpitts circuit (drawn to the right) is another well-known three-pole oscillator configuration. Note that because the inverting input of the opamp is a virtual ground the resistor \( r \) effectively shunts \( C_2 \). Assume an idealized opamp and as a customary simplification \( C_1 = C_2 \); verify that the transfer function is

\[
\frac{V_a}{V_b} = -G \left[ \frac{1}{(1 + \frac{R}{r}) (1 + s^2LC) + sC(2 + s^2LC + \frac{L}{CRr})} \right]
\]

(To derive the transfer expression note that the voltage across \( C_2 \) is \(-va/G_\), and work backwards to calculate \( vb \).)

The last term in the parentheses on the right in the denominator (coefficient of \( sCR \) ) ordinarily can be neglected (with capacitance values in microfarads, resistances in kilohms, and inductance in millehenries the order of magnitude of the term generally can easily be made much smaller than 2). It is not difficult to assure this design simplification over a wide range of oscillation frequencies, and in any event it is at least useful as a way of estimating the oscillation frequency. Within this approximation verify (explicitly) that oscillation occurs for \( \omega^2LC = 2 \), for an amplifier gain magnitude \( G \geq 1+ (R/r) \).

**Project:** Design a Colpitts oscillator for a nominal oscillation frequency of 10 kHz and nominal peak amplitude of 6 volts or more. Use \( L = 10 \text{ mH} \). Show explicitly how the individual Barkhausen conditions are met for your design and verify circuit performance using PSpice. Observe the amplifier voltage output, and compare it to the voltage across \( C_2 \). How does the improvement in waveform come about?

Note that the computer does not ordinarily provide thermal noise to initiate oscillation. Instead specify an initial voltage across the capacitor to provide the start-up energy.
(D) Hartley Oscillator

The oscillator circuit drawn to the right is the dual of the Colpitts oscillator; the roles of the inductor and the capacitors in the Colpitts circuit are interchanged to obtain the Hartley circuit. Assume an idealized opamp and, as is usual, \( L_1 = L_2 \); verify that the transfer function is

\[
\frac{v_3}{v_b} = -6 \left[ \frac{1}{(1 + \frac{R}{r})(1 + \frac{1}{s^2 LC}) + \frac{R}{sL} \left( 2 + \frac{1}{s^2 LC} + \frac{L}{C RT} \right)} \right]
\]

To derive the transfer expression note that the voltage across \( L_2 \) is \(-v_a/G\).

The last term in the parentheses on the right in the denominator ordinarily can be neglected (with capacitance values in microfarads, resistances in kilohms, and inductance in millehens the order of magnitude of the term generally can easily be made much smaller than \( 2 \)). It is not difficult to assure this design simplification over a wide range of oscillation, and in any event it is at least useful as a way of estimating the oscillation frequency. Within this approximation verify (explicitly) that oscillation occurs for \( \omega^2 LC = 1/2 \), for an amplifier gain magnitude \( G > 1 + (R/r) \). Note that the frequency of oscillation is half that of the Colpitts circuit for the same \( L \) and \( C \) values.

Design a Hartley oscillator for a nominal oscillation frequency of 10 kHz and a nominal peak amplitude of 6 volts or more. Use \( L_1 = L_2 = 10 \text{mH} \). Show explicitly how the individual Barkhausen conditions are met for your design and verify circuit performance using PSpice. Observe the amplifier voltage output, and compare it to the voltage across the tuned circuit. How does the improvement in waveform come about?

Note that the computer does not ordinarily provide thermal noise to initiate oscillation. Instead specify an initial voltage across the capacitor to provide the start-up energy.
**(E) Transfer Function Oscillator**

The active filter circuit drawn to the right provides another illustration of a circuit that can be made to oscillate. (The circuit is one used elsewhere to illustrate two-pole pulse response in general.) A straightforward transfer function analysis obtains the relationship

\[
\frac{v_b}{v_a} = G + j\omega RC + \frac{1}{j\omega RC} = G
\]

where \( \omega RC \geq 1 \)

\[ * \triangleq j\omega \]

Equating real and imaginary parts of the two sides of the equation obtains the two Barkhausen conditions \( G = 3 \) and \( \omega RC = 1 \).

Design an oscillator using this circuit for a nominal oscillation frequency of 10 kHz and nominal peak amplitude of 6 volts or more. Show explicitly how the individual Barkhausen conditions are met for your design and verify circuit operation using PSpice.

Note that the computer does not ordinarily provide thermal noise to initiate oscillation. Instead specify an initial voltage across the capacitor to provide the start-up energy.