Supplements Chapters 9 and 10.

APPENDIX D: ELASTIC BUCKLING

Long slender members may also exhibit excessive elastic deformations if the axial compressive force exceeds the critical buckling force (load). Thus care must be taken in design analysis to make certain that long slender members neither buckle nor yield.
Elastic Buckling of Long Slender Members:

To simplify the presentation we shall consider only elastic buckling in the plane of the paper. Then, the long slender straight member of Figure B.1 (Appendix B) may be simplified as in Figure D.1, where the elastic deflection of the member is also considered.

Static equilibrium analysis for the typical segment in Figure D.1(b) indicates that

\[
\frac{d(F_{xy})}{dx} + w_y(x) = 0
\]

and

\[
\frac{d(M_{xz})}{dx} + F_{xy} - F_{xx}(dv/dx) = 0
\]

Eliminating \( F_{xy} \) from these expressions we obtain

\[
\frac{d^2(M_{xz})}{dx^2} - F_{xx}(d^2v/dx^2) = w_y(x)
\]

Next, we shall substitute for \( M_{xz} \) in terms of the radius of curvature of the neutral axis, \( M_{xz} = EI_{yy}/R_{NA} \), and note that \( (1/R_{NA}) = + (d^2v/dx^2) \). Thus, equilibrium requires that

\[
EI_{yy}(d^4v/dx^4) - F_{xx}(d^2v/dx^2) = w_y(x)
\]

Specializing this result to the particular case where (a)
Figure D.1 Typical segment of a long slender initial straight member experiencing a lateral distributed force.
no distributed lateral load acts (viz., \( w_y(x) = 0 \)), and (b) \( F_{xX} \) is compressive and does not change along the entire length of the member, we may write

\[
d^4v/dx^4 + (P/EI_{yy})d^2v/dx^2 = 0 \quad [D.1]
\]

in which (+P) replaces (−\( F_{xX} \)), refer Figure D.2.

As evident in Figure D.2(b), the compressive axial force \( P \) generates a bending moment of magnitude \( (Pv) \) which tends to increase the lateral deflection \( v \). We call \( (Pv) \) the buckling moment, viz., it tends to cause the member to buckle. The buckling moment is resisted by the internal bending moment associated with the elastic deformation of the deflected member. We call \( M_{xz} = (EI_{yy}/R_{NA}) \) the restoring moment (viz., it tends to resist an increase in deflection and indeed would restore the member to the undeformed shape in Figure D.2(a) if the force \( P \) were removed from the member). If the applied force \( P \) exceeds some critical value, say \( P_{\text{crit}} \), the restoring moment is no longer adequate to resist further deflection and the member will buckle (viz., deflect further under no increase in \( P \)). On the other hand, as long as the applied force \( P \) does not exceed \( P_{\text{crit}} \), the restoring moment exactly balances the buckling moment and a unique relationship exists between \( P \) and \( v \).

Solving the fourth order differential equation [D.1], we write
Figure D.2  Geometry of interest (pinned-pinned ends) and first three buckling modes.
\[ v = c_1 + c_2 x \]  
\[ + c_3 \sin \left[ (P/EI_{YY})^{1/2} x \right] + c_4 \cos \left[ (P/EI_{YY})^{1/2} x \right] \]  

[D.2]

and note that four boundary conditions must be specified to establish the unique \( P, v \) relationship. These boundary conditions for the specific example illustrated in Figure to establish the unique \( P, v \) relationship. The boundary conditions evident by inspection of Figure D.2(a) are (assuming pinned ends):

1. \( v = 0 \) at \( x = 0 \)
2. \( M_{xz} = 0 = EI_{YY} (d^2v/dx^2) \) at \( x = 0 \)
3. \( v = 0 \) at \( x = L \)
4. \( M_{xz} = 0 = EI_{YY} (d^2v/dx^2) \) at \( x = L \)

Boundary conditions (1) and (2) are satisfied when

\[ (1) \quad 0 = c_1 + 0 + 0 + c_4 \]
and

\[ (2) \quad 0 = 0 + 0 + 0 - c_4 P \]
Hence, we see that (2) requires \( c_4 = 0 \), and in turn, (1) requires that \( c_1 = 0 \). Next, we consider boundary conditions (3) and (4), noting that \( c_1 = c_4 = 0 \). The relevant expressions are

\[
(3) \quad 0 = 0 + c_2 L + c_3 \sin[(P/EI_{YY})^{1/2}L] + 0
\]

and

\[
(4) \quad 0 = 0 + 0 - c_3 P \sin[(P/EI_{YY})^{1/2}L] + 0
\]

Boundary condition (4) is satisfied when either \( c_3 = 0 \) or

\[
(P/EI_{YY})^{1/2}L = 0, \pm \pi, \pm 2\pi, \ldots \text{, } \pm n\pi \quad [D.3(a)]
\]

In either case (3) is satisfied when \( c_2 = 0 \). Returning to (4) we observe that \( E \), \( I_{YY} \), and \( L \) are regarded as fixed and \( P \) is regarded as a parameter. If \( P \) is below the critical load, the member is straight (viz., \( c_3 = 0 \)). If \( P \) reaches the critical load, then

\[
(P/EI_{YY})^{1/2}L = \pm \pi
\]

or,

\[
P = P_{\text{crit}} = \pi^2 EI_{YY}/L^2 \quad [D.3(b)]
\]

which is termed the Euler buckling load (force). See Figure D.3.
Figure D.3 Long slender members should be designed such that neither yielding nor buckling occurs. (But because the actual member may initially be bent or subjected to bending caused by an eccentric axial load, the full stable region shown cannot safely be used in design. Rather, allowable loads should be computed using appropriate empirical expressions based on experimental data.)
If the member is restrained from buckling (in the first mode) under load $P_{crit}$, it will buckle (perhaps violently) under a higher load, say $P_2$ in Figure D.2.(d), where the deflection is illustrated for the second mode viz., where $n = 2$ in Equation [D.3(a)].

Exercises:

1. Find the critical buckling load for a member that is built-in at one end and free at the other end. Compare your result to the critical buckling load for the pinned-pinned example herein. [$P_{crit} = n^2\pi^2EI_{yy}/4L^2$]

2. Find the critical buckling load for a member that is built-in at each end. [$P_{crit} = 4n^2\pi^2EI_{yy}/L^2$]
Elastic Buckling of Thin Members:

Elastic buckling of thin members is beyond the scope of this text. Nevertheless, we explicitly note that elastic buckling is a potential mode of failure for any thin member that experiences local compressive stresses. For example, the floor joists in a typical house are restrained from lateral buckling by cross-bridging. The $2 \times 10^8$ (or $2 \times 10^5$) are selected in design to resist the bending moments associated with typical distributed floor loads, but are weak with regard to lateral bending (and buckling). The flanges on any fabricated I-beam, C-channel, box-section, etc., must be kept sufficiently small to prevent local buckling (kinking). The spacing of rivets and bolts must be restricted to prevent buckling of thin sections between the rivets or bolts, Figure D.4. Beads are used in sheet metal stampings and ribs are used in castings to provide lateral rigidity, etc.

In design we must learn to discern those proportions that are associated with service-proven members and components and to identify those proportions associated with problem members and components. The best design guides are service-proven designs. Theory allows us to extrapolate this design information from one application to another.
Figure D.4  Buckling between rivets in a stamped connection resisting a compressive load.