CHAPTER TEN

STRESS ANALYSIS FOR THE BENDING MOMENT COMPONENTS OF INTERNAL LOADS
Introduction

Most members and components experience bending. In this Chapter we shall examine the stress state associated with pure bending (i.e., where internal loads exhibit no shear force components).

First, we present the classical engineering mechanics (strength of materials) development of the normal stress distribution associated with bending of long straight slender members with uniform cross sections fabricated from a linear elastic homogeneous isotropic material. Then we examine analogous stress distributions when certain alternatives are of interest, e.g., curved members versus straight members, nonsymmetrical cross sections versus symmetrical cross sections, etc. Next, we present stress concentration factors for certain simple stress concentrators. And finally, we briefly discuss residual stress for bending.

The classical engineering mechanics approach to stress analysis has three basic steps: first, we hypothesize the strain distribution which not only appears reasonable, but also explicitly satisfies geometric compatibility, then we state the corresponding stresses using the assumed stress-strain relationships, and finally we assure that the resulting stress distribution satisfies static equilibrium. These analyses, while not always exact, suffice for numerous applications.
10.1

NOMINAL STRESS DISTRIBUTION FOR ELEMENTARY BENDING

When the given member is sufficiently long and slender that its longitudinal axis is obvious by inspection, then we may distinguish between bending and torsion, Figure 10.1. We shall stipulate that for elementary bending \( \bar{M}_{xy} = 0 \) and \( \bar{M}_{xz} \) is directed along the Z axis shown in Figure 10.2.

Now consider component geometry. For the symmetrical cross sections of Figure 10.2 only one coordinate distance, \( y \), is of special interest. Thus cartesian coordinates are appropriate, whatever the cross section shape. Next consider a typical segment of the member. In this elementary analysis we assume the following conditions define the typical segment:

1. Pure Bending - constant along the X axis
2. Uniform Cross Section - No change along X
3. Homogeneous Material - No \( X,Y,Z \) variation

Thus any segment with vertical (planar) sides is a typical segment, Figure 10.3(a). Geometric compatibility arguments dictate that the deformed segment must also have planar sides whose traces converge radially to the center of curvature of the neutral axis. This geometric compatibility information may restated in terms of strains, viz., Figure 10.3 indicates that

\[ e_{x1} = 0 \quad (= e_{xy} = e_{xz}) \]

and Figure 10.4 shows that
Figure 10.1 Distinction between bending and torsion. The longitudinal axis of the long slender member is parallel to the X axis.
Figure 10.2 Example symmetrical cross sections required for elementary analysis.

Note: These cross sections are symmetrical about the vertical Y axis. The bending moment component acts in the Z direction.
Figure 10.3  (Caption attached)

(a) Undeformed Member (No X Variation)

Typical Segment (Planar Sides)

(b) Deformed Member

Typical Deformed Segment (Radial Planar Sides)

(c) Typical Deformed Segment
Caption to Figure 10.3

(a) Typical segment in undeformed member.

(b) Same typical segment in deformed member. The deformed segment is defined by the arc $\delta \theta$. The relationship between $\delta \theta$ and $\ell$ is apparent in (c).

(c) When the typical segment deforms as in (b) the top portion (above a certain $y$ coordinate value, say $y^*$) is compressed whereas the bottom portion (below $y^*$) is stretched. There is no (zero) deformation at $y^*$. Thus it is termed the neutral axis (not surface). The local radius of curvature of the neutral axis is denoted $R_{\text{NA}}$. Since there is no $X$ variation, $R_{\text{NA}}$ is invariant along the length of the member. Namely, the deformed member in (b) bends in the arc of a circle and the deformed typical segment is defined by radial lines converging to the center of this circle. Accordingly, it is evident that $R_{\text{NA}} \delta \theta = \ell$. 
Figure 10.4  Development of $e_{xx} = -\frac{y}{R_{NA}}$.

Distance $a-a = R_{NA} \delta \theta = l_{\text{undeformed}}$

Distance $b-b = R \delta \theta = (R_{NA} - y) \delta \theta = l_{\text{deformed}}$

$$e_{xx} = \frac{l_{\text{deformed}} - l_{\text{undeformed}}}{l_{\text{undeformed}}}$$

$$e_{xx} = \left(\frac{R_{NA} - y}{R_{NA}}\right) \delta \theta - \frac{R_{NA} \delta \theta}{R_{NA}}$$

$$e_{xx} = -\frac{y}{R_{NA}}$$
\[ e_{xx} = -\frac{y}{R_{NA}} \]

Thus the only undefined strains at present are \( e_{yy} \), \( e_{zz} \) and \( e_{yz} \) (which must be symmetrical about the Y axis).

Next consider the stress-strain relationships. The generalized Hooke's law expressions indicate that

\[
S_{xy} = G e_{xy} = 0 \\
S_{xz} = G e_{xz} = 0 \\
S_{yz} = G e_{yz}
\]

and

\[
e_{xx} = -\frac{y}{R_{NA}} = \frac{1}{E}[S_{xx} - \nu(S_{yy} + S_{zz})] \\
e_{yy} = \frac{1}{E}[S_{yy} - \nu(S_{xx} + S_{zz})] \\
e_{zz} = \frac{1}{E}[S_{zz} - \nu(S_{xx} + S_{yy})]
\]

in which \( S_{xx} \), \( S_{yy} \), \( S_{zz} \), and \( S_{yz} \) are unknown at present.

We must now eliminate some of these unknowns before we can proceed with the development of the stress distribu-
bution for $S_{xx}$. Consequently we assume (by analogy with known stress distribution throughout a tension specimen) that $S_{yy} = S_{zz} = S_{yz} = 0$. Then, we may write

$$e_{xx} = -\frac{v}{E} S_{xx} = -\frac{Ey}{R_{NA}}$$
$$e_{yy} = e_{zz} = -\frac{v}{E} S_{xx}$$
$$S_{yy} = S_{zz} = 0$$
$$e_{yz} = 0 = S_{yz}$$

Our problem now is to make sure that $S_{xx}$ satisfies static equilibrium.

Equilibrium Analysis:

Consider Figure 10.5. Static equilibrium requires that

$$(\sum \vec{F}) \cdot \hat{\mathbf{x}} = 0 = \int S_{xx} dA = -\frac{E}{R_{NA}} \int y dA$$

and

$$(\sum \vec{M}) \cdot \hat{\mathbf{z}} = 0 = -M_{xz} - \int y S_{xx} dA$$

The first equation is satisfied when $y$ is measured from the centroid of the area $A$. The second equation dictates that

$$M_{xz} = +\frac{E}{R_{NA}} \int y^2 dA = +\frac{EI_{yy}}{R_{NA}}$$

in which $I_{yy}$ is the moment of inertia of the cross sectional area. Now substituting for $R_{NA}$
Figure 10.5  Static equilibrium conditions for a member that experiences elementary (pure) bending along its length.

(Free Body Diagram)

\[ \sum \vec{F} = \vec{0} = \int (S_{xx} \, dA) \hat{x} \]

\[ \sum \vec{M} = \vec{0} = -M_{xz} \hat{z} + \int (y \hat{y}) x (S_{xx} \, dA \hat{x}) \]

\[ = -M_{xz} \hat{z} - \int (y S_{xx} \, dA) \hat{z} \]

Note: \( \int (z \hat{z}) x (S_{xx} \, dA \hat{x}) = \vec{0} \) for symmetrical cross sections when \( z \) is measured from \( \bar{z} \), the \( Z \) centroid of the cross sectional area. See Figure 10.7.
\[ M_{xz} = -\frac{E I_{yy}}{E y / S_{xx}} \]

Summarizing in convenient form and using notation \( M_{xz} \) and \( I_{yy} \) to emphasize the use of centroidal axes

\[ S_{xx} = -\frac{M_{x Zy}}{I_{yy}} \quad [10.1(a)] \]

This stress distribution is plotted in Figure 10.6. The neutral axis lies at the Y centroid of the cross sectional area. The stress is compressive above the neutral axis, tensile below. (Note that the sign assumed for \( e_{xx} \) is consistent with the direction of the applied moment when the sign convention associated with double subscript notation is employed. From a rigorous perspective, the sign of the moment dictates the sign of the stress (equilibrium analysis) which in turn dictates the sign of the strain.)

Figure 10.7 is presented to show why a symmetrical cross section is assumed in elementary bending stress analysis.

Exercise:

Show that the sign of the imposed moment (Figure 10.5) dictates that \( S_{xx} \) is negative for positive \( y \), and that this is consistent with the sign of the assumed strain. Suppose however that \( M_{xz} = 0 \) and \( M_{x y} \neq 0 \), do you detect a sign problem? [Hint: Consider the plan view of Figure 10.16.]
Figure 10.6 Nominal normal stress distribution associated with elementary bending, viz., pure bending of homogeneous members with uniform symmetrical cross sections.
Figure 10.7 Sketch to show the necessity of the restriction of elementary bending stress analysis to symmetrical cross sections.

\[(\sum \vec{M}) \cdot \vec{r} = 0 = \int (z) S_{xx} \, dA \quad \text{satisfied by inspection for symmetrical cross sections}\]
Exercises:

1. Given a bar being bent, is the stress different for aluminum than steel. What material behavior assumptions are required.

2. Given the cross sections (a) through (e) below, assume that $M_{zz} = 200$ Newton-meters. Compute $S_{xx,\text{max}}$ for each cross section and rank the respective maximum stresses from smallest to largest according to cross section.

3. Consider cross sections (a), (b), (d), and (e). Size these cross sections such that each has 576 square mm area and that (b) and (e) have 6 mm walls. Rank the cross sections relative to their respective maximum bending stresses.

4. Given 6 mm flanges and webs, and a cross sectional area of 576 mm, state dimensions for (c) such that it just barely excels the best cross section in Exercise 3. above.
Comparison to the Elasticity Solution:

In defense of the preceding engineering mechanics solution for bending stresses in symmetrical members we note that: (1) the stress state

\[ S_{xx} = -\frac{M_{xz}y}{I_{yy}} \quad S_{yy} = 0 = S_{zz} = S_{xy} = S_{yz} = S_{zx} \]

satisfies the differential equations of equilibrium for cartesian coordinates, (2) the strain state

\[ e_{xx} = -\frac{M_{xz}y}{EI_{yy}} \quad e_{yy} = e_{zz} = +\frac{M_{xz}y}{EI_{yy}} \]
\[ e_{xy} = e_{yz} = e_{zx} = 0 \]

satisfies the compatibility equations (i.e., these strains are linear in \( y \) and the equations of compatibility are second order differential equations), and (3) the stress state above satisfies the boundary conditions relevant to the lateral surfaces of the member, viz., no shear or normal stresses acting on the lateral exterior (free) surfaces.

To satisfy the boundary conditions at the ends of the member requires that these ends be loaded by linear pressure distributions whose statical equivalents are the applied moments, viz., identical to the stress distribution \( S_{xx} \). This method of load application of course is completely impractical. Thus we must restrict our belief in the distribution of \( S_{xx} \) to regions outside of the Saint Venant zones.
at each end of the member, e.g., as illustrated in the photoelasticity results displayed in Figure 10.8.

The proof that the engineering mechanics solution is indeed the same as the exact (unique) elasticity solution is left to advanced texts. But it should be noted that the exactness of the nominal bending stress distribution is far less important in design stress analysis than subsequently using the correct stress concentration factor in analysis of real members. Namely, the expression \( S_{xx} = -\frac{M_{x}}{I_{yy}} \) by itself, exact or approximate, has little if any application. Its design application is almost always in conjunction with the appropriate stress concentration factor.
Photoelasticity exhibit of a long slender member with a uniform rectangular cross section experiencing pure bending along its nominal stress region (under the action of four-point loading).

Figure 10.8 Region of application of nominal bending stress expression. Note: If a hole were drilled in the center of this member, there would be an additional stress concentration region.
Anticlastic Curvature:

We have referred to the neutral axis throughout our presentation of the classical engineering mechanics solution for elementary bending, whereas it may seem tempting to refer to the neutral surface. It turns out that the neutral surface displays double curvature. It not only displays the radius of curvature $R_{NA}$ in Figure 10.4, but as shown in Figure 10.9, it exhibits a transverse curvature, viz., anticlastic curvature, which for the rectangular cross section illustrated is easily shown to be equal to $R_{NA}/\nu$. Thus the neutral axis is strictly speaking the only line lying on the neutral surface whose curvature is in the plane of symmetry of the cross section.
Figure 10.9  Anticlastic curvature of the neutral surface associated with elementary bending.

Deformed cross section shape

Undeformed rectangular cross section assumed for convenience in analysis

Sketch of the cross section of a member experiencing elementary bending, viz.,

\[ S_{XX} = - \frac{M_{XZ} v}{I_{YY}} \]
Exercises:

1. Examine the shape of the deformed cross section in Figure 10.9 and argue analytically that the deformed geometry shown is correct and that the anticlastic curvature is indeed equal to $\frac{R_{NA}}{\nu}$.

2. Examine the sketch of the deformed cross section in Figure 10.9 and state why the deformed geometry does not violate static equilibrium.

3. Suppose the member in Figure 10.9 actually had a round (circular) cross section before deforming. Sketch its cross section after deforming. What is the anticlastic radius of curvature.
THE GENERAL CASE

Suppose we now consider relaxing some of the restrictions on our nominal elementary bending stress analysis. These restrictions are summarized in Table 10.1, along with selected practical alternative cases. From one perspective the elementary bending stress analysis may be considered as a special case of the general analysis incorporating all alternative cases simultaneously. However, this general analysis is far too tedious to permit development herein (or any other text). Instead we merely examine below certain special cases of particular academic or design interest.

SPECIAL CASES (Refer Table 10.1)

1. Curved Members

The longitudinal strain \( e_{ee} \) at radius \( R \) in a typical segment of a long slender curved (symmetrical) member is, Figure 10.10,

\[
e_{ee} = \frac{(R-R_{NA})d\phi}{R\phi} = \frac{d\phi}{\phi}(1 - \frac{R_{NA}}{R})
\]

Also, by geometric compatibility arguments

\[
e_{er} = 0 = e_{rz} = e_{ze}
\]
Table 10.1  Summary of assumptions for elementary bending stress analysis, and some practical alternative assumptions

Engineering Mechanics
(Strength of Materials)
Solution

1. GEOMETRY

<table>
<thead>
<tr>
<th>Straight</th>
<th>Curved</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uniform Cross Section</td>
<td>Tapered</td>
</tr>
<tr>
<td>Symmetrical Cross Section</td>
<td>Nonsymmetrical</td>
</tr>
</tbody>
</table>

2. MATERIAL

<table>
<thead>
<tr>
<th>Elastic</th>
<th>Yielding</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$S_Y = S_Y^c$</td>
</tr>
<tr>
<td></td>
<td>$S_Y^t = S_Y^c$</td>
</tr>
<tr>
<td>Linear</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Linear $E_t \neq E_c$</td>
</tr>
<tr>
<td>Homogeneous</td>
<td>Nonlinear</td>
</tr>
<tr>
<td>Isotropic</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Laminated</td>
</tr>
</tbody>
</table>

3. LOADING

<table>
<thead>
<tr>
<th>Pure Bending</th>
<th>Nonuniform Bending (viz., with nonzero shear force components of internal loads)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Superimposed Axial Forces (viz., with nonzero axial force components of internal loads)</td>
</tr>
</tbody>
</table>
Figure 10.10 The typical segment for a curved member under pure bending is delimited by radial lines before and after deformation.
The orthogonal normal strains \( e_{rr} \) and \( e_{zz} \) are at present undefined.

From Hooke's law (with \( S_{rr} = S_{zz} = 0 \))

\[
S_{ee} = \frac{Ed\phi}{\partial e} (1 - \frac{R_{NA}}{R})
\]

and

\[
S_{er} = S_{rz} = S_{ze} = 0
\]

Note that as for straight members we again assume the normal stresses \( S_{rr} \) and \( S_{zz} \) are zero. This assumption clearly satisfies the obvious boundary conditions. Moreover, all stresses and strains are now stated explicitly.

Next consider static equilibrium analysis. By simple analogy with the bending stress analysis for a straight member,

\[
(\Sigma \vec{F}) \cdot \vec{e} = 0 = \frac{Ed\phi}{\partial e} \int (1 - \frac{R_{NA}}{R})dA
\]

and, evaluating moments about the center of curvature

\[
(\Sigma \vec{M}) \cdot \vec{z} = 0 = -M_{ez} + \frac{Ed\phi}{\partial e} \int (1 - \frac{R_{NA}}{R})RdA
\]

The first equation is satisfied when the neutral axis is located according to the expression

\[
R_{NA} = \frac{A}{\int \frac{dA}{R}}
\]
The second equation requires that

\[ M_{ez} = \frac{Ed\phi}{\delta \theta} [A(\bar{R} - R_{NA})] \]

in which \( \bar{R} \) is the radius to the centroid of the area, i.e., \( \frac{1}{A} \int R dA \). Substituting for \( (d\phi/\delta \theta) \) we obtain

\[ S_{ee} = + \frac{M_{ez}(R - R_{NA})}{AR(\bar{R} - R_{NA})} \]

in which \( (\bar{R} - R_{NA}) \) is positive, but \( (R - R_{NA}) \) is negative at the inside radius of the member. See Figure 10.11.

Exercises:

1. Given a rectangular cross section with inside radius \( r_i \), outside radius \( r_o \), and width \( b \); show that \( \int (1/R) dA = b \log_e (r_o/r_i) \).

2. Given the values \( M_{ez} = 200 \) Newton-meters, \( b = 24 \text{ mm} \), \( r_i = 60 \text{ mm} \), and \( r_o = 90 \text{ mm} \), plot \( S_{ee} \).

3. Given the values in 2. above, state the maximum error in \( S_{ee} \) assuming the elasticity expression in Chapter Nine is exact. What is the maximum error in \( S_{RT} \)? What is the error in the location of the neutral axis.
Figure 10.11  Schematic of $S_{ee}$ bending stress distribution in a curved member.
<table>
<thead>
<tr>
<th>Cross Section</th>
<th>( \int \frac{dA}{R} )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Rectangle</strong></td>
<td>( b \ln \frac{r_o}{r_i} )</td>
</tr>
<tr>
<td><strong>Circle</strong></td>
<td>( 2\pi \left( r_i + \frac{h}{2} \right) - \left[ \left( r_i + \frac{h}{2} \right)^2 - \frac{h^2}{4} \right]^{\frac{1}{2}} )</td>
</tr>
<tr>
<td><strong>Trapezoid</strong></td>
<td>( \frac{b_1 r_o - b_2 r_i}{h} \ln \frac{r_o}{r_i} - b_1 + b_2 )</td>
</tr>
<tr>
<td><strong>L-section</strong></td>
<td>( b_1 \ln \frac{r_i + h_1}{r_i} + b_2 \ln \frac{r_o}{r_i + h_1} )</td>
</tr>
<tr>
<td><strong>Modified I-beam</strong></td>
<td>( b_1 \ln \frac{r_i + h_1}{r_i} + b_2 \ln \frac{r_o - h_3}{r_i + h_1} + b_3 \ln \frac{r_o}{r_o - h_3} )</td>
</tr>
</tbody>
</table>
1a. Curved members bent perpendicular to the plane of the paper.

Suppose the curved member above experienced a bending moment $M_{er}$ instead of $M_{eZ}$. Then the normal strain experienced by the typical filament, Figure 10.12, is

$$e_{ee} = -\frac{z d\varphi}{(\hat{R} + r^*) \delta \theta}$$

The associated normal stress is thus (assuming $S_{rr} = S_{zz} = 0$)

$$S_{ee} = -\frac{E z d\varphi}{(\hat{R} + r^*) \delta \theta}$$

Static equilibrium requires that

$$(\sum \vec{F}) \cdot \delta = 0 = \int_A S_{ee} dA$$

$$(\sum \vec{M}) \cdot \hat{R} = 0 = \int_A (\hat{R} + r^*) S_{ee} dA$$

and

$$(\sum \vec{M}) \cdot \hat{R} = 0 = -M_{er} - \int z S_{ee} dA$$

The first two equations are satisfied for symmetrical cross sections (about the $\hat{R}$ axis) when $z$ is measured from the centroid, viz., when the neutral axis passes through the $z$ centroid of the cross section. The third equation requires that
Figure 10.12 Transverse bending of curved members.

Typical filament

(a) Plan view of the deformed typical segment. Note the radial sides of the segment form the angle $d\phi$.

(b) Front view of typical segment

Undeformed Filament Length equals $(\bar{R} + r^*)\delta\theta$
\[ M_{er} = \frac{Ed\phi}{\tilde{R}^2} \int_A \frac{z^2}{1 + r^*/\tilde{R}} \ dA \]

Let the integral be denoted \( Q_{zz} \). Then, substituting for \( \frac{d\phi}{\theta} \) we obtain the bending stress expression

\[ S_{ee} = -\frac{M_{er} \ z}{Q_{zz}[1 + r^*/\tilde{R}]} \]

in which

\[ Q_{zz} = \int_A \frac{z^2}{1 + r^*/\tilde{R}} \ dA \]

Note that if \( \tilde{R} \) becomes very large compared to the thickness of the member, the solution above reduces to the elementary bending stress solution derived for straight members.

Exercises:

1. Verify the equilibrium conditions above. Sketch the free body analogous to that in Figure 10.6 and sketch the symmetry constraint similar to that of Figure 10.8.

2. Given that \( M_{er} = 200 \) Newton-meters, \( b = 24 \) mm, \( r_1 = 60 \) mm, and \( r_0 = 90 \) mm, plot \( S_{ee} \) for this rectangular cross section.
2. Tapered Members

Figure 10.13 displays a sheet "constant-stress" specimen commonly used in fatigue testing. The specimen is loaded as shown so that the bending moment increases linearly towards the grip. But the specimen width also increases linearly towards the grip. Thus, assuming that the elementary bending stress analysis is valid, the bending stress along the tapered portion of the specimen is uniform (constant), i.e., independent of location.

Now consider Figure 10.14. Here we examine a small segment taken from the edge of a tapered sheet specimen. Suppose we assume that $S_{xx}$ acts as shown in (a) and may be computed using the elementary bending stress expression developed previously. It is clear that the top, bottom, and outside surfaces of the tapered segment are free of both normal and shear stresses. Thus the segment in (a) cannot be in static equilibrium unless other stresses such as displayed in (b) also act. But these stresses cannot be computed using the classical engineering mechanics approach. (We can only say that these stresses "should" be small compared to $S_{xx}$ if the taper is small.) Thus engineering judgment (and perhaps additional information) is required in the analysis of tapered members.

The "cantilever loading" indicated in Figure 10.13 is clearly not statically equivalent to pure bending, but it is common practice to use the elementary bending stress expression nevertheless. The validity of this practice is considered in Chapter 11.
Figure 10.13  "Constant-Stress" Fatigue Specimen

"Uniform Stress" Region

"Cantilever Loading"  
(Crank-driven deflection oscillates between fixed limits.)
Figure 10.14 Small segment taken from the edge of a tapered sheet specimen.

![Diagram of small segment](image)

Note: $S_{xy}$ and $S_{zy} \approx 0$ for sheet specimens (thin members).
Exercise:

Argue that $S_{xy}$ and $S_{zy}$ are negligible in Figure 10.14(b). Can any other stresses be ignored and still satisfy equilibrium conditions.
3. Nonsymmetrical Members

In Figure 10.15 we consider the case of a member with a rectangular cross section experiencing an arbitrary pure bending moment. In this elementary example we can compute the bending moment components in the Y and Z coordinate directions and then in turn compute the stress $S_{xx}$ at a particular point by algebraically adding the two $S_{xx}$ stresses computed using the respective bending moment components (and appropriate area moments of inertia). Suppose however the principal axes of the cross section were not apparent by inspection. Then we would first have to establish the orientation of the principal axes (say by Mohr's circle analysis) before we could state the required bending moment components in the superposition method of analysis.

Another, more direct, analysis is developed below. Suppose that, instead of establishing the orientation of the principal axes, say $X', Y', Z'$, we merely examine the state of stress for a member with an arbitrary uniform cross section, using the given coordinate axes, $X,Y,Z$. Refer sketch (b). The stress at a particular point $y,z$ shall be written as

$$S_{xx} = a + by + cz$$

where constants $a, b, c$ will be established by static equilibrium analysis.

Static equilibrium (refer Figure 10.16) requires that

$$\sum \vec{F} \cdot \hat{\mathbf{x}} = 0 = \int_{A} S_{xx} \, dA$$
Two methods of analysis for members with nonsymmetrical cross sections. (Use of both methods is advised to check the results.)

(a) Superposition Method.
When the principal axes are known (by analysis or inspection), $S_{xx}$ may be computed by considering the respective bending moment components separately in independent analyses, and then adding the results algebraically at each point on the cross section of interest.

Principal axes: $X,Y,Z$

(b) General Method.
Work with the $X,Y,Z$ coordinate axes (and the slightly more tedious bending stress expression developed herein) regardless of the shape of the cross section.

Principal Axes: $X',Y',Z'$
Coordinate Axes: $X,Y,Z$
Figure 10.16 Free body diagram for a long slender straight member with a nonsymmetrical cross section experiencing an arbitrary pure bending moment $\mathbf{M}_{xb}$ along its length.
and

\[
(\Sigma \vec{M}) \cdot \vec{x} = 0 = -M_{xy} + \int_{A} zS_{xx} \, dA
\]

\[
(\Sigma \vec{M}) \cdot \vec{y} = 0 = -M_{xz} - \int_{A} yS_{xx} \, dA
\]

Substituting for \( S_{xx} \) in the first equilibrium equation gives

\[
0 = a \int_{A} dA + b \int_{A} ydA + c \int_{A} zdA
\]

This equation is satisfied when constant \( a \) is set equal to zero and distances \( y \) and \( z \) are measured from the centroid of the cross sectional area. (If the member also experiences an internal axial force component \( F_{xx} \) acting through the centroid of the cross sectional area, the constant \( a \) would take on the value \( F_{xx}/A \).)

Next, substituting \( S_{xx} = by + cz \) into the second and third equilibrium equations gives (where \( \int_{A} \alpha \beta \, dA = I_{\alpha \beta} \))

\[
M_{xy} = bI_{yz} + cI_{zz}
\]

and

\[
-M_{xz} = bI_{yy} + cI_{yz}
\]

Treating these two expressions as two equations in two unknowns, namely \( b \) and \( c \), leads to the result

\[
S_{xx} = \left[ -\frac{M_{xz}I_{zz} - M_{xy}I_{yz}}{I_{yy}I_{zz} - I_{yz}^2} \right] y + \left[ \frac{M_{xy}I_{yy} + M_{xz}I_{yz}}{I_{yy}I_{zz} - I_{yz}^2} \right] z \quad [10.1(b)]
\]
in which the subscripts on the moment components and on the area moments and products of inertia are capitalized to emphasize the fact that these quantities pertain to centroidal axes.

Note that the general solution reduces to the elementary solution when \( I_{YZ} = 0 = M_{XY} \). If only \( I_{YZ} = 0 \), the general solution is equivalent to the superposition solution, Figure 10.15(a).

Exercise:

Given a long slender member with the uniform (angle iron) cross section, find the maximum bending stress and its location using both methods of analysis described above. Be sure to consider points A, B, C, D and E (as well as any other points that may be germane).
Angle Iron
(Ignore Radii & Fillets)

\[ \mathbf{M}_{xb} = -120\hat{y} - 120\hat{z} \]
Newton-meters
4. Yielding with Equal Yield Strengths in Tension and Compression (Elastic Perfectly Plastic Material), Rectangular Cross Section

Figure 10.17 displays the stress distribution for a long slender straight member with a uniform rectangular cross section when yielding has occurred to a depth of \( y^* \), assuming an elastic perfectly plastic material with equal yield strengths in tension and compression. Note that in this elementary example the neutral axis lies at the centroid of the cross section.

The moment required to cause yielding to a depth of \( y^* \) is

\[
M_{xz} = \frac{S_{YY} I_{YY}}{(h/2)} [1 + 2(y^*/h) - 2(y^*/h)^2]
\]

The maximum moment occurs when yielding extends all the way to the neutral axis (viz., when \( y^* = h/2 \)), and is termed the collapse moment, \( M_C \). Its value is equal to \( 3/2 M_e \), the maximum moment associated with purely elastic behavior.

Exercises:

1. Verify the expression given for \( M_{xz} \) pertaining to yielding of a member with a rectangular cross section.

2. Develop expressions for \( M_e \) and \( M_C \). Verify that their ratio is \( 3/2 \) for a rectangular cross section.
Figure 10.17  Stress distribution for a member with a rectangular cross section when yielding has occurred to a depth of $y^*$ (assuming an elastic perfectly plastic material with equal yield strengths in tension and compression).
We now consider briefly the curvature of the member as a function of applied bending moment.

Bending Moment-Curvature Relationship:

The form of the bending moment-curvature relationship appears in Figure 10.18. Here we use the symbol \( \rho (=R_{NA}) \) to denote the radius of curvature of the bent member. In turn, curvature is defined as \((1/\rho)\), i.e., the curvature is zero when the member is straight.

The bending moment-curvature relationship is developed as follows. First, recall that regardless of the material stress-strain relationship (Figure 10.5)

\[
e_{xx} = -\frac{y}{R_{NA}}
\]

for elementary bending of long slender members with uniform cross sections. Thus we may write

\[
\frac{1}{\rho} = \frac{1}{R_{NA}} = -\frac{e_{xx}}{y} = -\frac{e_y}{\frac{h}{2}y^*}
\]

which may be rewritten as

\[
\frac{y^*}{h} = \frac{1}{2} - \frac{e_y\rho}{h}
\]

Then, substituting this expression for \((y^*/h)\) into the expression for \(M_{xz}\) pertaining to yielding to depth \(y^*\) gives the bending moment-curvature relationship
Figure 10.18 Curvature versus bending moment. Note the spring back when the member is unloaded. (The ratio $M_c/M_e = 3/2$ pertains only to rectangular cross sections.)
\[ M_{xz} = \frac{S_Y I_{yy}}{(h/2)} \left[ \frac{3}{2} - \frac{e_y \rho}{h} \right] \]

This relationship may be re-expressed in more convenient form by introducing the reference relationship

\[ \left( \frac{1}{\rho} \right)_Y = - \frac{e_y}{h/2} \]

Then,

\[ M_{xz} = \frac{S_Y I_{yy}}{(h/2)} \left[ \frac{3}{2} - \frac{1}{2} \left( \frac{1}{\rho} \right)^2 \right] \]

Another convenient expression for the bending moment-curvature relationship is

\[ M_{xz} = \frac{3}{2} M_e \left[ 1 - \frac{1}{3} \left( \frac{1}{\rho} \right)^2 \right] \]

This expression was used to plot Figure 10.18.

Figure 10.18 shows the change in curvature associated with elastic spring back when the member is unloaded. Assuming the elastic perfectly plastic material model is valid, the member could be bent in the opposite direction with a moment \( M_{xz} = \frac{1}{2} M \) before yielding would occur upon "unloading".
Exercise Set One:

1. Calculate the bending moment required to cause yielding to the extent that $y^* = h/4$. Calculate also the curvature of the bent member after unloading and the spring back (change in curvature) during unloading.

2. For prior yielding such that $y^* = h/4$, calculate the magnitude of the bending moment applied in the opposite sense to the prior moment, which again causes yielding. Why does this value differ from $M_e$?

3. The bending moment-curvature relationship developed above is valid only for a specific range of curvature. State this range explicitly. State when this constraint was first implicitly introduced.
Exercise Set Two:

1. For the circular cross section below show that

\[
M_{xz} = \frac{S_y}{3} \sqrt{(r_o^2 - r_y^2)^3} + \frac{S_yr_o^2}{2} \sqrt{(r_o^2 - r_y^2)} \\
+ \frac{S_yr_o^4}{2r_y} \sin^{-1}\left(\frac{r_y}{r_o}\right)
\]

for an elastic perfectly plastic material with equal yield strengths in tension and compression.

2. Show that \(M_c / M_e\) for this cross section is equal to \(16/3\pi (\approx 1.7)\).
4a. ** Spring Back Temper Tester

Figure 10.19 presents a photograph of a spring back temper tester. This simple device is used to measure the yield strength of sheet (as a quality control check) by measuring the spring back of a strip of material after being bent around a 25mm diameter mandrel. The greater the spring back, the greater the yield strength. The advantages of the spring back temper tester are primarily: (a) the strip specimen is rectangular (25mm by 150mm) and therefore fabricated in seconds using mechanical shears, and (b) the device is small and easily portable from place to place in the rolling mill.

Operation: (1) The strip is fabricated and clamped into place. (2) The strip is bent 180 degrees about a 25mm diameter mandrel. (3) The strip is then allowed to spring back, and the spring back angle is read from a protractor attached to the spring back temper tester. (4) The spring back angle is used as an entry to a special chart (based on the analysis developed below) to obtain the yield strength of the material.

Analysis: Consider Figure 10.20. The moment associated with yielding to depth $y_Y$ is

$$M_{xz} = \frac{wt^2}{4} S_Y - \frac{wy_Y^2}{3} S_Y$$
Spring Back Temper Tester. (The yield strength of sheet stock is often loosely described in terms of its "temper").

Cross section of test strip specimen, with equal yielding assumed at the top and bottom surfaces.
This moment may be related to the specimen curvature using the general expression (Figure 10.5)

\[ \frac{1}{\rho} = -\frac{e_{xx}}{y} \quad \frac{1}{\rho} = \frac{1}{R_{NA}} \]

in which the sign indicates that the material toward the center of curvature is in compression. The strain at \( y = y_Y \) is the yield point strain \( e_Y \). The corresponding stress is \( S_Y/E \) (assuming \( S_{yy} \) and \( S_{zz} = 0 \)). Thus, if we denote the radius of the mandrel as \( R_M \) and assume that \( R_M + t/2 \approx R_M \),

\[ \frac{1}{R_M} = -\frac{S_Y}{Ey_Y} \]

Solving for \( y_Y \) we obtain

\[ y_Y = -\frac{S_Y R_M}{E} \]

Now let the radius of curvature after spring back be denoted \( R_F \). The decrease in curvature associated with spring back is (refer Figure 10.18)

\[ \frac{1}{R_M} - \frac{1}{R_F} = \Delta(\frac{1}{\rho}) \]

in which \( \Delta(\ ) \) infers change in ( ). And for elastic unloading

\[ \Delta(\frac{1}{\rho}) = \frac{\Delta(M_{xz})}{EI_{yy}} \]
in which (a) the magnitude of the change in moment associated with unloading is equal to the applied moment $M_{xz}$, and (b) the sign of the unloading moment is such that the radius of curvature increases, viz., $R_F > R_M$. Thus we write

$$\frac{1}{R_M} - \frac{1}{R_F} = \frac{wS_Y}{E I_{yy}} \left[ \frac{t^2}{4} - \frac{y_Y^2}{3} \right]$$

and substitute for $y_Y$ to obtain (after some algebra)

$$1 - \frac{R_M}{R_F} = 3K - 4K^3$$

where $K$ is defined as

$$K = \left( \frac{R_M S_Y}{E t} \right).$$

The term $(R_M/R_F)$ may be related to the spring back angle by noting the arc length is invariant, i.e., $rde = \text{constant} = (R_M + t/2)(\pi/2) \approx R_M(\pi/2)$. Thus,

$$\frac{R_M}{R_F} = \frac{\pi}{2} - \phi$$

Or,

$$1 - \frac{R_M}{R_F} = \frac{\phi}{180} = 3K - 4K^3$$
Exercises:

1. Prepare a chart so that the yield strength may be read from the appropriate curve given the specimen thickness (in the range from 4mm to 10mm) and the spring back angle in degrees. Assume that $E$ equals 210 Gpa, $D_M = 25\text{mm}$, and that the yield strength of interest will fall in the range, 240 to 540 Mpa.

2. Suppose you are concerned with the error introduced by the approximation $R_M + t/2 \approx R_M$. Rework the analysis symbolically and correct the curves in Exercise 1 above.
Exercise:

Consider the winding of a helical compression spring using round wire. The spring O.D. after spring back is sufficiently larger than the wound O.D. that the arbor must be sized to take spring back into account. Let $R_A$ equal the arbor radius and $R_F$ equal the inside radius of the spring after spring back. Show that

$$\frac{R_A}{R_F} = 1 - \frac{2}{\pi}[\sin^{-1}(2K) + \frac{2}{3}(K)(5SK^2)\sqrt{1 - 4K^2}]$$

where $K$ is defined as

$$K = \frac{R_A S_Y}{2E r_w}$$

and $r_w$ is the radius of the round wire. State all assumptions explicitly.
5. Yielding with Unequal Yield Strengths in Tension and Compression (Elastic-Perfectly Plastic Material)


7. Nonlinear Stress-Strain Relationship

These three cases each involve using the same fundamental methodology. Figure 10.21 shows the stress distribution relevant to both unequal yield strengths and unequal elastic modulii. The critical issue here is simply that the bending stress distribution is dictated by (a) the assumed strain distribution, and (b) the assumed stress-strain behavior. Two parameters describe the assumed linear strain distribution, e.g., $e_{aa}$ and $e_{bb}$, the normal strains at the top and bottom surfaces of the bent member. These two parameters must be chosen such that the two scalar equations of static equilibrium, $(\sum F) \cdot \bar{X} = 0$ and $(\sum M) \cdot \bar{Z} = 0$, are satisfied. The analytical problem is that the algebra of solution becomes quite tedious unless massive simplification occurs, for example when $E_C$ equals $E_T$, $S_{Y,C}$ equals $S_{Y,T}$ etc.

From a design perspective it appears that the bending stress analysis should be viewed as a numerical problem unless a simple algebraic solution is readily available. In this context, all bending stress analyses are numerical, except for a few well known special cases. Accordingly we outline below a simple trial and error procedure which is easily programmed on a digital computer.
Figure 10.21  Assumed linear strain distribution, assumed stress-strain relationship, resulting stress distribution.

(a) Assumed linear strain distribution

(b) Assumed stress-strain relationship

(c) Resulting stress distribution
Solution Method:

The details of numerical analysis will depend on the experience of the programmer and the software available. If the stress-strain relationship is given in graph form, it must be digitized. But even if digital stress-strain data are available, some procedure must be established to interpolate between specific entries. The details of these steps can range from very crude to quite elaborate. But, generally speaking with efficient programming, precise elaborate algorithms are not significantly more costly than crude less precise algorithms.

Figure 10.22 presents the skeleton of a flow chart for a computer program. Essentially it involves two loops, each associated with iteratively satisfying one of the scalar equations of static equilibrium. The acceptable error in (approximately) satisfying these two equations depends both on the grid size used in analysis and the accuracy of the given stress-strain data. The programmer must exercise judgment in establishing this accuracy.
Figure 10.22  Skeleton of a flow chart for a numerical bending stress analysis.

INPUT

Set $e_{aa}, e_{bb}$

Compute
\[ \int S_{xx} dA \]

Does $\int S_{xx} dA = 0$?

Adjust $e_{bb}$

Yes

Compute
\[ \int yS_{xx} dA \]

Does $\int yS_{xx} dA = M_{xz}$?

Adjust $e_{aa}$

OUTPUT
Exercise Set One:

Write a digital computer program to estimate $S_{xx,\text{max}}$. Assume a rectangular cross section and use existing computer software for numerical integration. Let $b = 24\text{mm}$, $h = 24\text{mm}$, and $M_{xz} = 200$ Newton-meters. Assume that the stress-strain diagram is the same in compression as in tension.

1. Let $S_{xx} = 150e^{0.25}_{xx}$ Gpa.

2. Let $S_{xx} = 110e^{0.25}_{xx}$ Gpa.

3. Let the actual stress-strain relationship be approximated by the stress-strain diagram below.

![Stress-Strain Diagram](image-url)
Exercise Set One-A:

Extend the program written in Exercise Set One to cover cross sections of the form (b) through (e) page 10.14.

1. Compare the result of Exercise Set One with that pertaining to (b) with 576 square mm cross sectional area and 6mm walls.

2. Compare the result of Exercise Set One with that pertaining to (d) with 576 square mm cross sectional area.

3. Compare the result of Exercise Set One with that pertaining to (e) with 576 square mm cross sectional area and a 6mm wall thickness.
Exercise Set Two:

Write a digital computer program to locate the neutral axis and to estimate both \( S_{xx, \text{max}}, \text{tension} \) and \( S_{xx, \text{max}}, \text{compression} \). Assume a rectangular cross section and use existing computer software for numerical integration.

Let \( b = 24\text{mm} \), \( h = 24\text{mm} \), and \( M_{xz} = 200 \text{ Newton-meters} \).

1. Let \( S_{xx, t} = 150e^{0.25} \text{ Gpa} \) and \( S_{xx, c} = 165e^{0.14} \text{ Gpa} \).

(Verify numerically that

\[
y_{NA} = \frac{b}{1 + \sqrt{\frac{E_t}{E_c}}}\]

and

\[
S_{xx, t} = -\frac{3M_{xz}y}{bh^3}[1 + \sqrt{\frac{E_t}{E_c}}]^{2}
\]

\[
S_{xx, c} = -\frac{3M_{xz}y}{bh^3}[1 + \sqrt{\frac{E_t}{E_c}}]^{2}\left(\frac{E_c}{E_t}\right)
\]

2. Let \( S_{xx, t} = 150e^{0.25} \text{ Gpa} \) and \( S_{xx, c} = 240e^{0.14} \text{ Gpa} \).

3. Let the actual stress-strain relationship be approximated by the stress-strain diagram below.
Exercise Set Two-A:

Extend the program written in Exercise Set Two to cover cross sections of the form (b) through (e) page 10.14.

1. Compare the result of Exercise Set Two with that pertaining to (b) with 576 square mm cross sectional area and 6mm walls.

2. Compare the result of Exercise Set Two with that pertaining to (d) with 576 square mm cross sectional area.

3. Compare the result of Exercise Set Two with that pertaining to (e) with 576 square mm cross sectional area and a 6mm wall thickness.
Exercise Set Three: **

1. Digitize the stress-strain diagram below and extend the program written in Exercise Set Two to cover this data input situation.

2. Suppose that the bent member has a nonsymmetrical cross section. Write a digital computer program which estimates the bending stress at a particular point on the cross section. Use the given X,Y,Z coordinate axes.
8. Lamellar Members

For purposes of illustrating the essential concepts of the analytical solution, we shall assume the member has only two lamellae, oriented perpendicular to the plane of symmetry of the long slender straight member. This analysis may subsequently be generalized to $N$ such lamellae.

Equilibrium Analysis:

Static equilibrium requires that

$$\int_{A} S_{xx} dA = 0 = \int_{A_1} S_{xx} dA_1 + \int_{A_2} S_{xx} dA_2$$

and

$$M_{xz} = -\int_{A} y S_{xx} dA = -\int_{A_1} y S_{xx} dA_1 - \int_{A_2} y S_{xx} dA_2$$

The first equilibrium equation is satisfied when

$$0 = \frac{E_1}{R_{NA}} \int_{A_1} y dA_1 + \frac{E_2}{R_{NA}} \int_{A_2} y dA_2$$

Substituting $(Y - Y_{NA})$ for $y$ and evaluating the integrals we obtain

$$0 = E_1(\bar{Y}_1A_1 - Y_{NA}A_1) + E_2(\bar{Y}_2A_2 - Y_{NA}A_2)$$
Figure 10.23 Symmetrical cross section of lamellar member with lamella perpendicular to the plane of symmetry.
Solving for $Y_{NA}$ gives

$$Y_{NA} = \frac{E_1 A_1 \bar{y}_1 + E_2 A_2 \bar{y}_2}{E_1 A_1 + E_2 A_2}$$

Or, in general,

$$Y_{NA} = \frac{\Sigma E_i A_i \bar{y}_i}{\Sigma E_i A_i}$$

The second equilibrium equation is satisfied when

$$M_{xz} = \frac{1}{R_{NA}} [E_1 I_{YY_1} + E_2 I_{YY_2}]$$

But,

$$R_{NA} = -\frac{V}{e_{xx}} = -\frac{E_i y}{S_{xx_i}}$$

Hence,

$$M_{xz} = -\frac{S_{xx_i}}{E_i y} [E_1 I_{YY_1} + E_2 I_{YY_2}]$$

Re-expressing in terms of $S_{xx_i}$ we write
\[ S_{xx1} = - \frac{M_{xz}^Y}{I_{YY_{eqv.}}} \]

in which the equivalent area moment of inertia is defined as

\[ I_{YY_{eqv.}} = \frac{\sum E_i I_{YY_i}}{E_i} \]

Exercise Set One:

Locate the neutral axis and compute \( S_{xx, max,t} \) and \( S_{xx, max,c} \). Assume the cross section is 24\text{mm} by 24\text{mm} with 6\text{mm} flanges and web. Let \( M_{xz} = 200 \) Newton-meters.

1. \[ \begin{array}{c} E = 210 \text{ Gpa} \\ E = 84 \text{ Gpa} \\ E = 210 \text{ Gpa} \end{array} \]

2. \[ \begin{array}{c} E = 84 \text{ Gpa} \\ E = 210 \text{ Gpa} \\ E = 84 \text{ Gpa} \end{array} \]

3. \[ \begin{array}{c} E = 210 \text{ Gpa} \\ E = 84 \text{ Gpa} \\ \text{E= 120 Gpa} \end{array} \]
Exercise Set 2:

1. Consider the circular geometry above. Find the respective maximum values of $S_{xx}$. (Suppose the outer member were shrunk on the inner member. How would this change the analysis.)
2. Consider a reinforced concrete member with a rectangular cross section. Assume that the concrete below the neutral axis cracked when the member was bent and thus carries no tensile stress, viz., assume that the steel reinforcement (below the neutral axis) carries the entire tensile stress. Estimate $S_{xx,\text{max},c}$ in the concrete if the applied bending moment $M_{xz} = 45,000$ Newton-meters, $E_{\text{steel}} = 210$ Gpa, and $E_{\text{concrete}}$ is approximately equal to $1000S_U$, where $S_U$, the ultimate strength of the concrete in compression, is about 21 Mpa. What is the naive factor of safety. What yield strength is required for the steel to have the same value for the naive factor of safety. (Suppose both materials have the same naive factor of safety. Are they equally safe. If not, what is the value of a naive factor of safety.)

![Cross section through reinforced concrete member](image)

Area Reinforcing = 1200$mm^2$

Steel
9. Nonuniform Bending

The bending moment changes along the length of a member when the internal load diagrams display nonzero shear force components. Nevertheless it is generally assumed that the elementary bending stress expression (for pure bending) is still valid. We shall develop expressions in Chapter 11 that indicate this assumption is correct provided that the shear force components do not change along the length of the member, but approximate otherwise. The transverse shear stress $S_{xy}$ cannot be assumed equal to zero in this case however, and the analysis developed in Chapter 11 must be used to estimate $S_{xy}$. (In Chapter 13 we consider the problem of establishing the local maximum shear stress $S_{ij}$ considering both $S_{xx}$ and $S_{xy}$.)
10. Superimposed Axial Forces

Consider the schematic of a heavy-duty crane hook in Figure 10.24. The hook experiences both bending and an axial force component at its critical cross section. The stress at any point on this critical cross section may be computed using superposition, viz., \( \sigma_{nn} = \sigma_{ee} + P/A \), where \( \sigma_{ee} \) is estimated by analysis of a round curved member experiencing the appropriate bending moment.

Exercises:

1. Suppose in Figure 10.24 the critical cross section is 24 mm in diameter and the e is approximately 36mm (i.e., \( r_i = 24 \text{mm} \)). What load (force) \( P \) will cause a normal stress of 240 Mpa at Point b. What is the corresponding stress at Point a.

2. ** Let the critical cross section in Figure 10.24 be trapezoidal. Using the notation of Table 10.2, let \( r_i = 24 \text{mm}, h = 24 \text{mm}, \) and \( S_{xx,max,c} = 1.12S_{xx,max,t} = 400 \text{ Mpa}, \) establish dimensions \( b_1 \) and \( b_2 \) such that the cross section has minimum area. Assume \( P = 50,000 \) Newtons.
Figure 10.24 Schematic of a heavy-duty crane hook. Load (force) \( \vec{F} \) causes both an axial stress and an elementary bending stress. These two normal stresses add algebraically. \( S_{mn,\text{max}} \) is tensile and occurs at Point b.

Internal Loads at Section a-b:

\[
\vec{F}_{ee} = \left( \frac{P}{A} \right) \delta
\]

\[
\vec{M}_{ez} = -(eP) \hat{t}
\]
STRESS CONCENTRATION FACTORS

Most members and components are stepped rather than tapered. Thus the engineering mechanics expression for nominal bending stress finds design application when it is appropriately modified to account for stress concentration. Figure 10.25 pertains to elementary bending of long slender members with circular cross sections. It presents stress concentration factors for fillets and grooves.

Exercises:

1. Given the example internal load diagrams below, find the local maximum normal stress associated with bending. Consider each groove to find the critical section.

2. If the yield strength of the material in tension is 350 Mpa and the yield strength of the material in compression is 520 Mpa, compute the naive factor of safety.

3. If a naive factor of safety equal six is desired, size each throat diameter such that the same value for the local maximum normal stress occurs at each groove.
Example Internal Load Diagrams
Figure 10.25 Principal stress concentration factors for long slender straight members with circular cross sections, experiencing a pure bending moment $M_{xb}$. $K_{nn \text{nom}} = M_{xb} (d/2)/\left(\pi d^4/64\right)$.

(a) Grooves

(b) Fillets
RESIDUAL STRESS

Figure 10.26 shows the stress distribution in a bent steel bar with a rectangular cross section determined by X-ray (strain measurement) techniques: (a) with yielding (compare Figure 10.17), and (b) the corresponding residual stress distribution developed upon unloading. Figure 10.27 in turn shows the analogous theoretical stress distributions based on an analysis employing the elastic perfectly plastic material model. Clearly the agreement is quite good. Nevertheless, the credibility of the elastic perfectly plastic modeling is diminished by the observed discontinuous (heterogeneous) yielding associated with Piobert-Lüders bands. Figure 10.28.
Figure 10.26

(a) Stress distribution under yielding. Applied moment $\bar{M}_{xz}$ taken as negative to correspond to sign in Figure 10.26.

(b) Elastic unloading stress distribution. Unloading moment negates applied moment $-\bar{M}_{xz}$.

(c) Residual stress distribution (a) - (b).

No applied moment.

Figure 10.27 Analytical development of the residual stress distribution associated with yielding of an initially stress-free member with a rectangular cross section by overstraining in elementary bending.
Figure 10.28

Discontinuous (heterogeneous) yielding during bending of a steel bar with a rectangular cross section. The observed depth of yielding by Piobert–Lüders bands agrees remarkably well with the calculated depth based on the applied moment $\bar{M}_{xz}$ and an assumed elastic perfectly plastic material behavior. Reference: Smith and Sidebottom, Elementary Mechanics of Deformable Bodies, The Macmillan Co., 1969. See Figure 7.15.

Note: Yielding occurs on 45 degree planes, i.e., Piobert–Lüders band AB is oriented at 45 degrees on the front face of the specimen, while band EF is oriented at 45 degrees on the top of the specimen.
Figure 10.29 Final curvature for the bent bar in Figure 10.28 after unloading. The associated residual stress distribution has the opposite sign of that appearing in Figures 10.26 and 10.27. (The bar must be yielded in bending in the opposite direction exactly the right amount to straighten it.)
Exercises:

1. Explain how the heavy-duty crane hook in Figure 10.24 can be overloaded as the final step in manufacturing to increase its naive factor of safety in subsequent service operation.

2. Calculate the maximum tensile value of residual stress for the bar in Figure 10.27 assuming that the bar was yielded to a depth of \( h/4 \) before it was unloaded.

3. ** Sketch the residual stress distribution for the bar in Figure 10.27 after it has been straightened by appropriate bending.

4. **** Compute the magnitude of the bending moment required to straighten the bar in Figure 10.27 assuming that the bar was yielded to a depth of \( h/4 \) before it was unloaded.

5. **** Assume that the material has different yield strengths in tension than compression but is otherwise elastic perfectly plastic. Derive the residual stress distribution analytically.
WELDED MEMBERS

The state of stress in a fillet weld is so complex that welds are sized on the basis of elementary calculations coupled with allowable stresses stated by appropriate Codes (which are based on experience and test data). We shall now describe the elementary weld stress calculations that are analogous to the elementary bending stress calculations.

Elementary Analysis:

Consider the welded bracket in Figure 10.30. Define the basal plane of the fillet welds as shown. If the bracket is loaded through the hole, the statical equivalent force and moment acting at the centroid of the weld pattern in the basal plane causes the bracket to tend to translate and rotate. The coordinate moment components that lie in the basal plane are associated with bending because the theoretical effect of these components is to generate normal stress distributions in the welds (on the basal plane) that are analogous to the normal stress distribution associated with elementary bending, compare Figures 10.7 and 10.31. In the latter diagram we note that assuming the bracket is rigid-like implies a linear strain distribution, which in turn implies a linear stress distribution for a Hookean material. Static equilibrium analysis then proceeds as developed previously for elementary bending except that $y$ is measured from the $Y$ centroid of the weld pattern and $I_{YY}$ pertains to the area moment of inertia of the welds about the centroidal axis.
Figure 10.30 Example welded bracket, with the basal planes of fillet welds identified.

Basal Plane for 6mm welds

Basal Plane for 3mm welds

16-42

6

25-75
Figure 10.31 Theoretical normal stress distribution acting on the weld areas in the basal plane developed assuming that the bracket is rigid-like.
Exercises:

1. Criticize the elementary theoretical analysis from both a practical and theoretical point of view. First, consider the difference between the actual and assumed geometries. Then, consider the stress distribution assumed for Section A-A in Figure 10.31. Does it seem reasonable that the stress is uniform across the fillet weld.

2. Compute the maximum values for the normal stresses $S_{nn}$ pertaining to the welded details below. If your answer does not agree with that given, rework your analysis changing your assumptions until you obtain the answer given.

(a) $M_{nt}$ (t=2w)

$$S_{nn, \text{max}} = - \frac{M_{nt}}{6Lt^2}$$

(b) $M_{nt}$ (t>2w)

$$S_{nn, \text{max}} = - \frac{3M_{nt}t}{wL} \sqrt{\frac{1}{3t^2 - 6tw + 4w^2}}$$

(c) $M_{nt}$ (t=2w)

$$S_{nn, \text{max}} = - \frac{M_{nt}}{6t^2}$$
(d) \[ S_{nn, \text{max}} = -\frac{3M_{nt}}{wl^2} \quad (t > 2w) \]

(e) \[ S_{nn, \text{max}} = -\frac{M_{nt}}{wl(t+w)} \]

(f) \[ S_{nn, \text{max}} = -\frac{3M_{nt}}{wl^2} \]

(g) \[ S_{nn, \text{max}} = -\frac{4M_{nt}}{\pi wD^2} \]
3. Compute the maximum values of the normal stresses $S_{nn}$ pertaining to the welded brackets given in Sketches One and Two. Start your stress analysis by stating the statically equivalent force and moment acting at the centroid of the weld areas in the basal plane. Two components of the moment theoretically generate stress distributions that are analogous to elementary bending stress distributions. Let $P = 1500$ Newtons and $\theta = 30$ degrees unless stated otherwise.

(a) Sketch One
(b) Sketch Two, 3mm fillet welds
(c) Sketch Two', 6mm fillet welds
Sketch One

28mm x 28mm x 3mm Angle Iron
BOLTED MEMBERS

Bolted (and riveted) members are statically indeterminate. Thus we must make certain assumptions regarding the (ideal) geometry to obtain a solution for the shear and normal forces acting on the conceptual cross sections of the bolts at the basal plane of the bracket or member of interest. Generally the assumption is made that the bracket is rigid-like. Then, in Figure 10.32, we can allege that as the bracket tends to rotate due to the statically equivalent moment associated with the action of the cross pin on the bracket, the resulting tensile deformations are proportional to the ratio \( r_f/r_x \). Hence, for a Hookean material, we allege that the bolt (screw) normal stresses associated with the specific rotation shown also differ by the ratio \( r_f/r_x \). Our analysis then requires that these bolt stresses (forces) and the associated action of the base plate on the bracket satisfy

\[
\Sigma (\vec{F}_{fbb}^* + \vec{F}_{rbb}^* + \vec{F}_{bp}^*) \cdot \hat{N} = 0
\]

and

\[
\Sigma (\vec{F}_{fbb}^* + \vec{F}_{rbb}^* + \vec{F}_{bp}^* + \vec{M}_{z,se}) \cdot \hat{l} = 0
\]

in which the bolt (screw) head forces act (only) downward, and the base plate force acts (only) upward. The modeling in Figure 10.32 pertains to bolts (screws) that are only "finger tight", viz., not pretensioned.
Figure 10.32 Bolted joint example. Each view is analyzed separately (as for welded points) and then the solutions are added using "superposition".

Angle Iron Bracket -- Side View

Z Component of Statically Equivalent Moment Associated with the Action of the Cross Pin on the Bracket

\[ \vec{F}_{\text{Base Plate on Bracket Corner}} \]

\[ \vec{F}_{\text{Front Bolt Head on Bracket}} \]

\[ \vec{F}_{\text{Rear Bolt Head on Bracket}} \]

\[ \delta \theta \] (small)
When the bolts (screws) are adequately pretensioned the contact between the base plate and the bracket is modeled as occurring in the immediate vicinity of the bolts (screws), Figure 10.33. Then, when the bracket tends to rotate at the hookeye acts on the cross pin (Figure 10.32) and the cross pin acts on the bracket, the clamping pressure is assumed to decrease under the front bolt head and increase under the rear bolt head. Specifically, the center of rotation is taken as the centroid of the bolt pattern because the decrease in bolt tension may be viewed as being statically equivalent to a "compressive" force.

The credibility of the modeling is limited whether it is assumed the bolts (screws) are "finger tight" or adequately pretensioned. Thus, either analysis only provides a starting point in seeking alternative analyses which more accurately reflect the physics of the actual situation. Unfortunately, for bolted joints (experiencing tensile rather than shear stress) there are no Codes which provide allowable stresses based on experience and testing to circumvent the problems we encounter in estimating the actual bolt (screw) stresses. This means that we must expend greater effort to estimate the actual service loads and we must also carefully consider all reasonable (although perhaps somewhat extreme) alternative loads and models because failure seldom occurs under the ideal conditions assumed in elementary analyses. It is also good design practice to use a larger factor of safety when relevant information is lacking.
Figure 10.33 Preliminary Modeling for Pretensioned Bolts (Screws).

(a) Angle Iron Bracket Fastened With Pretensioned Bolts (Screws). No Load Applied on Cross Pin by Hookeye (Figure 10.32).

(b) Loading Causes a Small Bracket Rotation and a Decrease in the Front Bolt (screw) Tensile Normal Stress. This Decrease in Tensile Stress is Statically Equivalent to an "Imposed" Compressive Stress. Hence, the Bracket Rotates About the Centroid of the Bolt (screw) Pattern Rather Than About a Corner as in Figure 10.32.
Exercise:

Suggest an alternative center of rotation for both the "finger tight" and the "adequately pretensioned" cases just discussed.

Exercises:

Compute the (total) normal tensile stresses in each bolt (screw) for the brackets given in Sketches Three and Four. Start your analysis by stating the statically equivalent force and moment acting at the centroid of the bolt (screw) pattern in the basal plane. Let $P = 1500$ Newtons and $\theta = 30$ degrees unless stated otherwise.

(a) Sketch Three
(b) Sketch Four
Sketch Four

6mm AISI C-1022
As Rolled Stock

M8 x 16 screws