CHAPTER ELEVEN

STRESS ANALYSIS FOR THE SHEAR FORCE COMPONENTS
OF INTERNAL LOADS
Introduction

The deformations associated with the shear force components of the internal loads acting in long slender members are too complex to be deduced by simple reasoning. Thus, the classical three-step engineering mechanics approach to stress analysis cannot be used here. Rather, we must adopt an indirect but effective approach in which we assume that the normal stress distribution associated with pure bending is also valid for nonuniform bending, and in which we specifically select a free body for static equilibrium analysis that includes a longitudinal shear stress $S_{yx}$ which may subsequently be equated to the transverse shear stress $S_{xy}$ of interest.

For long slender members the transverse shear stress associated with the shear force components of internal loads is usually negligible compared to the normal stress associated with bending. Thus, the engineering mechanics expression for transverse shear stress developed herein (Equation [11.1]) is primarily of academic interest relative to its use in designing against yielding. However, the concomitant concepts of shear flow and shear center have practical application in designs where open thin-walled sections are employed.
NONUNIFORM BENDING

The bending moment changes along the length of a member when the internal load diagrams, Figure 11.1, display nonzero shear force components. Equilibrium analysis shows that, at locations where the internal loads may be modeled as continuous, the local rate of change of the bending moment component is related to the corresponding local magnitude of the shear force component by

\[
\frac{d(M_{xz})}{dx} = -F_{xy}
\]

and

\[
\frac{d(M_{xy})}{dx} = +F_{xz}
\]

using the conventional X,Y,Z coordinate system (Appendix B).

Equilibrium Analysis:

For simplicity in presentation we shall restrict our introductory discussion to elementary bending of members with symmetrical cross sections. Consider a typical segment of the member, Figure 11.2(a). The normal stress associated with bending changes from one side of the typical segment to the other side because of a corresponding change in the bending moment. Thus the top portion of this typical segment, sketch (b), cannot be in static equilibrium unless the bottom portion of the typical segment acts on the top portion, imposing (at least) a longitudinal shear stress $S_{yx}$. Since the typical segment experiences no normal or shear stresses acting on its lateral surfaces,
Figure 11.1 Example Internal Load Diagrams
Figure 11.2  (a) Typical segment of a long slender straight member with a symmetrical cross section, experiencing nonuniform elementary bending \((M_{xy} = 0)\), and (b) top portion of the typical segment with longitudinal shear stress \(S_{yx}\) acting on the bottom conceptual surface whose area is denoted \(A_b\).

\[ M_{xz} = M_{xz}' = M_{xz} + \frac{d(M_{xz})}{dx}(\delta x) \]

\(S_{xx}\), stress distribution at location \(x\)

\(S_{xx}'\), stress distribution at location \(x + \delta x\)

Free Body Diagram

See Attached Note (page 11.4)
Note: Given a positive bending moment $M_{xz}$, the stress above the neutral axis is compressive. If, in addition, $d(M_{xz})/dx$ is also positive, the compressive stress acting on the positive $X$ face is larger in magnitude than the compressive stress acting on the negative $X$ face. In this case, the direction of the longitudinal shear stress $S_{yx}$, (b), is in the positive $X$ direction. But this stress acts on a negative $Y$ face; therefore $S_{yx}$ is negative. This negative sign is consistent with the sign of $F_{xy}$ for a positive $d(M_{xz})/dx$. Refer Figure 11.1. In the text discussion, however, we generate the correct (and consistent) signs by direct use of the sign convention associated with double subscript notation in our static equilibrium analysis for the free body diagram in (b).
\[(\sum \vec{F}) \cdot \vec{I} = 0 = \int_{A^*} S_{xx}' dA^* - \int_{A^*} S_{xx} dA^* - \int_{A^*} S_{yx} dA_b\]

in which double subscript sign convention is used (whereas the compressive normal stresses sketched indicate that bending moments \(M_{xz}'\) and \(M_{xz}\) are positive). This equation has no unique solution because the variation of \(S_{yx}\) over the bottom area \(A_b\) of the top portion of the typical segment is unknown. Suppose however, that we assume that \(S_{yx}\) displays negligible \(Z\) variation because the member is thin. Then, as \(\delta x\) approaches zero in the limit, we may write

\[S_{yx,ave} = \frac{\int_{A^*} \frac{d(M_{xz})}{dx} (dx)y + \int_{A^*} \frac{M_{xz}'}{I_{yy}} dA^*}{\int_{A^*} \frac{M_{xz}'}{I_{yy}} (dx)y} \]

in which \(A_b = t(\delta x) = tdx\), and \(S_{yx,ave}\) denotes the average value of \(S_{yx}\) (which is now assumed to have negligible \(Z\) variation). Simplifying this expression we next write

\[S_{yx,ave} = \frac{\int_{A^*} \frac{d(M_{xz})}{dx} y dA^*}{I_{yy} t}\]

and then using the bending moment–shear force relationship, \(d(M_{xz})/dx = -F_{xy}\), we obtain the result

\[S_{yx,ave} = \frac{F_{xy} \int_{A^*} y dA^*}{I_{yy} t}\]
We recommend rewriting this result using the notation

\[ S_{nx} = \frac{F_{xy} \int ydA^*}{I_{yy} t} \]  \[ \text{[11.1(a)]} \]

(Thin Members)

in which the thin member restriction (viz., the negligible variation in \( S_{nx} \) in the "thickness" (i.e., the short transverse) direction) is stated explicitly. But however the expression is written, must be careful that thickness \( t \) is measured in the appropriate direction. For example, consider a typical segment of an I-beam in elementary bending, Figure 11.3. Thickness \( t \) for the I-beam flanges corresponds to dimension \( t_1 \). The dimension \( b \) is not appropriate. As a second example, consider the thin-walled tube in Figure 11.4. In this case, thickness \( t \) is measured in the radial direction, regardless of the location of interest. The latter example is presented to emphasize the point that \( F_{xy} \) and \( I_{yy} \) depend only on the coordinate system used in internal load analysis (and subsequently to compute \( S_{nx} \) for elementary bending), whereas \( t \) and \( A^* \) depend on the coordinate system appropriate to the geometry at the location where we wish to compute \( S_{nx} \).

Exercise:

Reconsider the stresses acting on the bottom surface of the top portion of the typical segment, Figure 11.2(b). Argue that \( S_{yy} = S_{yz} = 0 \). When is the assumption of thin members helpful.
Figure 11.3  Shear stress $S_{zx}$ induced in flanges of I-beam under nonuniform elementary bending. Note that thickness $t$ infers dimension $t_1$ not $b$. 

(a) 

(b)
Figure 11.4  Thickness $t$ for a thin-walled tube is measured in the radial direction at each location of interest.
The Transverse Shear Stress $S_{xy}$:

So far we have only discussed the variation of $S_{yx}$ as a function of $y$ ($y_L$ in Figure 11.5). However, our primary interest lies in the examination of $S_{xy}$. These two shear stresses are equal in magnitude at each location of interest, as indicated by a simple static equilibrium analysis for an infinitesimal stress element taken from the intersection of the bottom surface and either end of the segment portion displayed in Figure 11.5(a). The appropriate infinitesimal element appears in (b). Sketch (c) in turn shows that $S_{xy}$ is parabolic (zero at the top and bottom and maximum at the neutral axis) for members with rectangular cross sections, viz.,

$$S_{xy} = S_{yx} = \frac{F_{xy} \int y \, (tdy)}{I_{yy} t} = \frac{F_{xy}}{2I_{yy}} \left[ (h/2)^2 - y_L^2 \right]$$

in which the upper limit $y_U$ equals $h/2$ and the lower limit $y_L$ is used as a parameter in plotting.

We may now use the results of the static equilibrium analysis of the stress element in Figure 11.5(b) (viz., $S_{ij} = S_{ji}$) to write (for the transverse shear stress $S_{xn}$)

$$S_{xn} = \frac{F_{xy} \int y \, dA^*}{I_{yy} t} \quad [11.1(a)]$$

(Thin Members)
Figure 11.5  

Relation between $S_{xy}$ and $S_{yx}$. Note that at $y = y_L$ the magnitude of $S_{xy}$ equals exactly the magnitude of $S_{yx}$. This $S_{ij} = S_{ji}$ relationship is evident from a static equilibrium analysis for the infinitesimal element in (b), which pertains to location $y_L$. In turn, plotting $S_{xy}$ as a function of $y_L$ it is clear that the transverse shear stress distribution is parabolic for members with rectangular cross sections that experience nonuniform elementary bending.
Exercises:

1. Extend Figure 11.5(b) by showing that $S_{xy, \text{max}}$ occurs at $y_L = 0$, and its value there is $(3/2)F_{xy}/A$.

2. Show that for the positive $Z$ face in Figure 11.3(b)

\[ S_{zx} = S_{xz} = -\frac{F_{xy} \int y dA^*}{I_{YY} t} \quad \text{(Thin Members)} \]

Note the sign for this expression.

Thick Members:

Consider the circular cross section in Figure 11.6(a). We shall now argue that the direction of the transverse shear stress depends (in general) on the lateral location of interest at distance $y_L$ from the neutral axis. In (b) we observe that static equilibrium conditions (viz., that $S_{ij} = S_{ji}$) require that the shear stress $S_{sx}$ acting on an infinitesimal stress element at the surface of the member must equal the transverse shear stress $S_{xy}$. But clearly $S_{sx} = 0$. Thus $S_{xy} = 0$. Accordingly, the only permissible shear stress is $S_{xe}$ in (c), acting parallel to the surface. Thus the direction of the shear stress changes along $y_L$ as sketched in (a). This complication can invalidate
Figure 11.6 Transverse shear stress direction and distribution for members with circular cross sections. Equation [11.1(a)] has severe limitations in this application.

S_{sx} = 0 at the surface. Therefore S_{xy} = 0 at the surface. See (c).

Only nonzero S_{xe} permissible at the surface.

(d) Elasticity solution for the shear stress distribution across the centerline of the round member. The corresponding engineering mechanics (strength of materials) solution (Equation [11.1(a)]) assumes a uniform distribution at each value of y_L.
the engineering mechanics expression for estimating the magnitude of the transverse shear stress. Accordingly, we restrict our analyses to situations where it is clear that the shear stress directions are parallel, e.g., where $y_L = 0$ for members with circular cross sections, and to members with rectangular cross sections.

Exercises:

1. The elasticity solution for the transverse shear stress distribution across the centerline of a round member experiencing nonuniform elementary bending is shown in Figure 11.6(d). Evaluate this shear stress using the engineering mechanics expression for $S_{yx, ave}$ and state the percentage error at both the vertical centerline and at each edge (where $r = r_0$).

2. Consider the equations of equilibrium for plane stress and verify that

$$\frac{\partial S_{xx}}{\partial x} + \frac{\partial S_{yx}}{\partial y} = 0$$

$$\frac{\partial S_{xy}}{\partial x} + \frac{\partial S_{yy}}{\partial y} = 0$$

when

$$S_{xx} = -\frac{M_{xz,y}}{I_{yy}}$$

and, given a rectangular cross section

$$S_{yx} = S_{xy} = \frac{F_{xy}}{2I_{yy}}[(h/2)^2 - y_L^2]$$
Note also that the second differential equation requires that \( S_{xy} \) is constant along \( X \).

3. In the development of the engineering mechanics expression for computing the magnitude of the transverse shear stress (assuming thin members) we assumed without comment that the engineering mechanics expression for pure bending could be used even when the bending is nonuniform. Use the information given in Exercise 2. above to critique the credibility of this assumption and the accuracy of the engineering mechanics expression.

4.** Sketch the shear deformation for a typical segment of a member experiencing nonuniform elementary bending, Figure 11.2(a). [Hint: Use your knowledge of the shear stress distribution.]

5.**** Consider the portion of the cross section of the thin-walled tube in Figure 11.4 from \( \theta = \theta_L \) to \( \theta = \theta_U \) and state expressions for the shear stresses acting on these two radial (internal) surfaces. Let \( \theta = 0 \) degrees pertain to the horizontal centerline of the tube cross section.
Nonsymmetrical Cross Sections:

The analysis of the transverse shear stress acting in members with nonsymmetrical cross sections is the same as for members with symmetrical cross sections, except that the algebra is slightly more tedious. We shall restrict our discussion here to thin members, and use the conventional \(X,Y,Z\) coordinate axes rather than principal axes. Then, the normal stress \(S_{xx}\) associated with bending is given by

\[
S_{xx} = \frac{-M_{xz}I_{ZZ} - M_{xy}I_{YZ}}{I_{ZZ}I_{YY} - I_{YZ}^2} y + \frac{M_{xy}I_{YY} + M_{xz}I_{YZ}}{I_{ZZ}I_{YY} - I_{YZ}^2} z
\]

Considering static equilibrium for a free body defined as a portion of a typical segment delimited by a positive \(N\) face upon which longitudinal shear stress \(S_{nx}\) acts, we write

\[
(\Sigma \vec{F}) \cdot \vec{a} = 0 = \int S_{xx}^t dA^* - \int S_{xx} dA^* + S_{nx}(t \delta x)
\]

Solving for \(S_{nx}\) and letting \(\delta x\) approach zero in the limit, we obtain

\[
S_{nx} = \int \frac{d(M_{xz})}{dx}(-I_{ZZ}y + I_{YZ}z) + \frac{d(M_{xy})}{dx}(-I_{YY}z + I_{YZ}y)}{t[I_{ZZ}I_{YY} - I_{YZ}^2]} \ dA^*
\]

Next, substituting for \(-d(M_{xz})/dx\) and \([d(M_{xy})/dx\) we write

\[
S_{nx} = \int \frac{F_{xy}(-I_{ZZ}y + I_{YZ}z) + F_{xz}(-I_{YY}z + I_{YZ}y)}{t[I_{ZZ}I_{YY} - I_{YZ}^2]} \ dA^*
\]
But we know that $S_{1j} = S_{ji}$. Thus, the transverse shear stress is

$$S_{xn} = S_{nx} = \int \frac{F_{xy}(-I_{ZZ} + I_{YY}z) + F_{xz}(-I_{YY}z + I_{YZ})}{A^*} \frac{dA^*}{t[I_{ZZ}I_{YY} - I_{YZ}^2]}$$

For simple cross sections, calculations are enhanced by using the partitioned expression

$$S_{xn} = S_{nx} = \int \frac{(-F_{xy}I_{ZZ} + F_{xz}I_{YZ})}{A^*} \frac{ydA^*}{t[I_{ZZ}I_{YY} - I_{YZ}^2]} \quad [11.1(b)]$$

(Thin Members)

$$+ \int \frac{(-F_{xz}I_{YY} + F_{xy}I_{YZ})}{A^*} \frac{zdA^*}{t[I_{ZZ}I_{YY} - I_{YZ}^2]}$$

and evaluating the integrals using the identities

$$K_1 \int ydA^* = K_1 \bar{y}A^* \quad \text{and} \quad K_2 \int zdA^* = K_2 \bar{z}A^*$$

See Figure 11.7.

Exercise:

Let $I_{YZ} = 0 = F_{zx}$ and verify that the sign of $S_{nx}$ agrees with the sign of $S_{yx}$, as developed earlier. Use a rectangular cross section to simplify your analysis.
Figure 11.7 Illustration of \( \bar{y} \), \( \bar{z} \), and \( A^* \) for the identities

\[
\int ydA^* = \bar{y}A^* \quad \text{and} \quad \int zdA^* = \bar{z}A^*
\]
SHEAR FLOW DIAGRAMS

Figure 11.8 shows the directions of the transverse shear stresses $S_{nx}$ associated with nonuniform elementary bending of common constructional members. The sketches in Figure 11.8 are called shear flow diagrams, where shear flow is defined as

$$\frac{d(F_{nx})}{dx} = q_{nx} = t(S_{nx}) \text{ Units: lbsf/in}$$

This definition is useful in certain situations where the thickness $t$ varies but the product $t(S_{nx})$ is invariant. The direction of the shear stress (shear flow) at any location of interest may be established directly using the partitioned engineering mechanics expression for $S_{nx}$ (Equation [11.1(b)]), which pertains to a positive $N$ face, and double subscript sign convention. However, to enhance physical understanding, the directions are deduced in two examples which follow using simple sketches and equilibrium oriented arguments.

In Figure 11.9 we analyze the shear flow in an I-beam that experiences known internal loads. We shall assume that $M_{xy} = 0, M_{xz} > 0$ and $d(M_{xz})/dx > 0$ for purposes of illustration. With this information we can establish the signs and relative magnitudes of $S_{xx}$ and $S'_{xx}$ (the values of normal stress acting on the negative and positive $X$ faces) at any location of interest and at any point on the cross section. In turn, this information suffices to state the direction of the net force associated with these two normal stresses. Hence, the direction of the shear stress $S_{nx}$ which satisfies static equilibrium is evident, which then dictates the
Figure 11.8 Shear flow diagrams for common constructional members showing the directions of the transverse shear stresses $S_{x n}$. These directions pertain to analyses in which it is assumed that $M_{xy} = 0$ and that $d(M_{xz})/dx > 0$. 
Figure 11.9 Development of shear flow diagram for an I-beam. Analysis assumes that $M_{xy} = 0$, $M_{xz} > 0$ and $d(M_{xz})/dx > 0$. (In and out refers to the $Y,Z$ plane, viz., the plane of the paper for the I-beam cross section.)

Segment a-a Top View

Segment b-b Side View

Segment c-c Top View
direction of the corresponding shear stress $S_{xn}$. For example, consider the upper right flange segment defined by a-a in Figure 11.9. Here we see that the compressive normal stress acting on the positive X face exceeds the compressive normal stress acting on the negative X face. Hence the net force associated with these two normal stresses is directed into the plane of the paper (towards the negative X face). Hence $S_{nx}$ acts out of the plane of the paper, which in turn dictates that $S_{xn}$ acts to the left on the positive X face. (If this direction is not clear, consider static equilibrium for an infinitesimal stress element located at the intersection of the N face and the positive X face. Note that $S_{xn}$ is zero at the extreme edge of the flange (because $A^*$ is zero) and increases linearly as a-a moves to the left (because $S_{xx}$ is constant and $A^*$ increases linearly with Z).

Next, consider segment b-b in Figure 11.9. The net force associated with the two normal stresses again acts into the plane of the paper. Thus shear stress $S_{nx}$ must act out of the plane of the paper, and $S_{xn}$ therefore acts downward. The magnitude of $S_{xn}$ increases (parabolically) as b-b moves closer to the neutral axis, reaches a maximum, and then decreases for b-b below the neutral axis because, although $A^*$ continues to increase, the sign of the normal stress changes from compressive to tensile and the corresponding net force decreases.

Exercises:

1. Verify the direction sketched for $S_{xn}$ pertaining to segment c-c in Figure 11.9.

2. Verify the direction of the shear flow for the C-channel in Figure 11.10.
Figure 11.10 Development of shear flow diagram for a C-channel. Analysis assumes that $M_{xy} = 0$, $M_{xz} > 0$ and $d(M_{xz})/dx > 0$. (In and out refers to the $Y,Z$ plane, viz., the plane of the paper for the C-channel cross section.)
3. Use the text methodology to establish the direction of the shear flow, given:

(a) $M_{xy} = 0$, $M_{xz} > 0$, and $d(M_{xz})/dx < 0$ in Figure 11.9.

(b) $M_{xy} = 0$, $M_{xz} < 0$, and $d(M_{xz})/dx > 0$ in Figure 11.9.

(c) $M_{xy} = 0$, $M_{xz} < 0$, and $d(M_{xz})/dx < 0$ in Figure 11.9.

(d) $M_{xy} = 0$, $M_{xz} > 0$, and $d(M_{xz})/dx < 0$ in Figure 11.10.

(e) $M_{xy} = 0$, $M_{xz} < 0$, and $d(M_{xz})/dx > 0$ in Figure 11.10.

(f) $M_{xy} = 0$, $M_{xz} < 0$, and $d(M_{xz})/dx < 0$ in Figure 11.10.

Does your shear flow diagram agree with that given in the text. Discuss.

4. Use the partitioned engineering mechanics expression (Equation [11.1(b)]) and double subscript sign convention to verify the direction of the shear flow given in

(a) Figure 11.9.

(b) Figure 11.10.

5. Use the partitioned engineering mechanics expression (Equation [11.1(b)]) and double subscript sign convention to establish the direction of the shear flow, given:

(a) $M_{xy} = 0$ and $d(M_{xz})/dx < 0$ in Figure 11.9.
(b) \( M_{xy} = 0 \) and \( \frac{d(M_{xz})}{dx} > 0 \) in Figure 11.9.

(c) \( M_{xy} = 0 \) and \( \frac{d(M_{xz})}{dx} < 0 \) in Figure 11.9.

(d) \( M_{xy} = 0 \) and \( \frac{d(M_{xz})}{dx} < 0 \) in Figure 11.10.

(e) \( M_{xy} = 0 \) and \( \frac{d(M_{xz})}{dx} > 0 \) in Figure 11.10.

(f) \( M_{xy} = 0 \) and \( \frac{d(M_{xz})}{dx} < 0 \) in Figure 11.10.

Does your shear flow diagram agree with that given in the text. Discuss.
Numerical Example:

To illustrate the numerical details of shear flow analysis we shall examine a C-channel and develop expressions pertinent to the given geometry which may subsequently be used to establish numerical values (and to locate the shear center SC in Figure 11.8). The dimensions of interest for the C-channel are given in Figure 11.11(a). Accordingly, centroid of the cross sectional area is located at

\[
\bar{z} = -\frac{2(at)(a/2)}{4at} = -\frac{a}{4}
\]

\[
\bar{y} = 0 \text{ by inspection}
\]

The area moments of inertia are

\[
I_{YY} = \frac{t(2a)^3}{12} + 2(at)a^2 = \frac{8}{3} ta^3
\]

\[
I_{ZZ} = 2\frac{t(a)^3}{12} + 2(at)(-\frac{a}{4})^2 + (2at)(\frac{a}{4})^2 = \frac{5}{12} ta^3
\]

and by inspection

\[
I_{YZ} = 0
\]

Starting at the upper right corner we note that along the top leg, \(\bar{y} = a, \bar{z} = -(3a/4) - (z_1/2)\), and \(A^* = t(z_1)\). Hence,

\[
S_{xn} = S_{nx} = -\frac{F_{xy}}{I_{YY}}(az_1) + \frac{F_{xz}}{I_{ZZ}}(\frac{3}{4}az_1 - \frac{1}{2}z_1^2)
\]
Figure 11.11 Example shear flow analysis for a C-channel.

(a) Specific geometry assumed in analysis.
The first term in this expression for $S_{xn}$ is linear, the second term is quadratic (parabolic). These two terms are plotted separately in Figures 11.11(b) and (c).

Next, along the vertical web, and deliberately choosing a segment bounded by a negative $Y$ face to illustrate a sign problem,

$$
\bar{y} = \frac{(at)a + (y_1t)(a - \frac{1}{2}y_1)}{at + y_1t} = \frac{a^2 + ay_1 - \frac{1}{2}y_1^2}{a + y_1}
$$

$$
\bar{z} = \frac{(at)(-\frac{1}{4}a) + (y_1t)(\frac{1}{4}a)}{at + y_1t} = \frac{-a^2 + ay_1}{4(a + y_1)}
$$

$$
A^* = at + y_1t
$$

But Equation [11.1] pertains specifically to a positive $Y$ (N) face. Therefore we must reverse the sign (or change about to work with a positive $Y$ face). Accordingly,

$$
S_{xn} = + \frac{F_{xy}}{I_{yy}}(a^2 + ay_1 - \frac{1}{2}y_1^2) - \frac{F_{xz}}{I_{zz}}[\frac{a}{4}(a - y_1)]
$$

Finally, along the lower leg, $\bar{y} = a$, $\bar{z} = -(\frac{5}{4}a - \frac{z}{2})$

and $A^* = tz_2$. Thus,

$$
S_{xt} = + \frac{F_{xy}}{I_{yy}}(az_2) + \frac{F_{xz}}{I_{zz}}[\frac{3}{4}az_2 - \frac{1}{2}z_2]
$$
Figure 11.11(b) Shear flow considering only the internal shear force component \( F_{xy} \).
\[ S_{xn} = + \frac{F_{xz}}{I_{ZZ}} (\frac{3}{4} a z_1 - \frac{1}{2} z_1^2) \]

\[ S_{xn} = - \frac{F_{xz} [a - y_1]}{I_{YY}} \]

\[ S_{xn} = + \frac{F_{xz}}{I_{ZZ}} (\frac{3}{4} a z_2 - \frac{1}{2} z_2^2) \]

Figure 11.11(c) Shear flow considering only the internal shear force component \( F_{xz} \).
Consider Figure 11.11(b). Here we let \( F_{xy} \) be
negative to correspond to the left hand portion of the
example internal load diagram in Figure 11.1 where, as in
Figure 11.8, \( d(M_{xz})/dx > 0 \). Then, \( S_{xn} \) is positive along the
top leg, i.e., directed to the left. But \( S_{xn} \) is negative
along the vertical web, i.e., directed downward. And, \( S_{xn} \)
is also negative along the bottom leg, i.e., directed to
the right.

Next consider Figure 11.11(c). Here we let \( F_{xz} \) be
negative to correspond to the left hand portion of the
example internal load diagram in Figure 11.1, where \( d(M_{xy})/dx \)< 0. In this case \( S_{xn} \) is negative along the top leg (the
opposite direction of \( S_{xn} \) in Figure 11.11(b)), as well as
along the bottom leg (the same direction as \( S_{xn} \) in Figure
11.11(b)). The sign of \( S_{xn} \) along the vertical web depends
on the location of interest. It is positive (directed upward)
above the midpoint and negative (directed downward) below.

Exercises:

1. Verify that the respective shear stress expressions
agree when evaluated at the intersection of each leg and
the web in both Figures 11.11(b) and (c).

2. Rework the analysis for the vertical web, this time
using a positive \( Y \) (N) face. Does your result agree with the
text.

3. State the disadvantages of using a negative \( N \) face in
analysis, viz., why is it poor practice.
We shall now check our analysis by integrating the shear stresses in each leg and along the web to obtain their static equivalents.

First, in Figure 11.11(b), along the top leg,

\[ \int_S S_{\text{xn}} \, dA = - \frac{F_{xy}}{I_{YY}} \left( \frac{a^3}{2} \right) t \]

whereas along the bottom leg

\[ \int_S S_{\text{xn}} \, dA = + \frac{F_{xy}}{I_{YY}} \left( \frac{a^3}{2} \right) t \]

Thus, \((\Sigma F_\bullet) \cdot \mathbf{\hat{z}} = 0 = - \frac{F_{xy}}{I_{YY}} (a^3/2)t + \frac{F_{xy}}{I_{YY}} (a^3/2)t\). (Checks)

Then, for the vertical web,

\[ \int_S S_{\text{xn}} \, dA = + \frac{F_{xy}}{I_{YY}} \left( 2a^3 + \frac{4}{2} a^3 - \frac{8}{6} a^3 \right) t = + \frac{F_{xy}}{I_{YY}} \left( \frac{8}{3} a^3 \right) t \]

But \(I_{YY} = \left( \frac{8}{3} ta^3 \right)\). Thus,

\[ \int_S S_{\text{xn}} \, dA = + F_{xy} \]

(and \(F_{xy}\) is negative in Figure 11.11(b)).
Next, in Figure 11.11(c), along each leg,

\[ \int_{A} S_{Yn} \, dA = + \frac{F_{xz}}{I_{zz}} \left( \frac{1}{6} a^3 - \frac{1}{24} a^3 \right)t = \frac{F_{xz}}{I_{zz}} \left( \frac{5}{24} a^3 \right)t \]

But \( I_{zz} = \frac{5}{12} ta^3 \). Thus, considering both legs,

\[ \int_{A} S_{Yn} \, dA = + \frac{F_{xz}}{I_{zz}} \]

(and \( F_{xz} \) is negative in Figure 11.11(c)).

Finally, for the vertical web,

\[ \int_{A} S_{Yn} \, dA = - \frac{F_{xz}}{I_{yy}} \left( \frac{a}{2} - \frac{a}{2} \right)^3 = 0 \]

This result, as well as each of the previous results, agrees with the applied loading assumed in the example.

Exercise:

Integrate the expression for the shear stress in the vertical web based on a positive Y face (Exercise 2, page 11.30). Does this result agree with the text.
Shear Center:

When estimating the axial normal stress \( S_{xx} = \frac{F_{xx}}{A} \), we always assume that \( F_{xx} \) (vector) acts at a specific location on the cross section, viz., the centroid of the cross sectional area. Otherwise, the member also experiences bending. Similarly, for nonuniform bending, we must assume that \( F_{xy} \) and \( F_{xz} \) act at a specific location relative to the cross section, viz., the shear center (denoted SC in Figure 11.12). Otherwise the member will twist as well as bend.

Examination of the shear flow diagrams in Figure 11.12 indicates that \( (\Sigma \hat{M}) \cdot \hat{x} = 0 \) when the moments associated with the shear stresses shown are evaluated about the respective shear centers. For the members illustrated, only the C-channel requires numerical computation to locate the shear center. Consider Figure 11.13. Sketch (a) shows the resultant shear forces in each leg and along the vertical web when only \( F_{xy} \) is considered as in Figure 11.11(b). Sketch (b) in turn shows a force \( F_{xy} \) that, if applied at the proper location, is statically equivalent to the force system in (a). For the dimensions assumed, this location is a distance \( 5a/8 \) to the left of the centroid of the cross sectional area. Examination of Figure 11.11(c) shows that the shear center lies on the horizontal centerline.

Figure 11.14 shows one way to load a C-channel member so that the member bends but does not twist. However, this claim is credible only in areas remote to the regions in which the member is loaded.
Figure 11.12 For nonuniform bending, the shear force components of the internal loads must act through the shear center (SC). Otherwise, the members will twist as well as bend.
(a) Shear forces statically equivalent to the shear stresses in Figure 11.11(b).

(b) Shear force statically equivalent to the shear forces in (a).

Figure 11.13 Sketches and static equivalents used to locate the shear center displayed in Figure 11.14.
Figure 11.14 Loading a C-channel through its shear center.
(Location 5a/8 pertains specifically to the dimensions of the text example.)
The design implication of shear flow and shear center is to avoid wherever possible twisting of members with open thin sections, just as we avoid long unrestrained flanges experiencing compressive normal stresses. Thus, when openings must be cut in members with closed sections, consideration should be given to the possibility that the result is an open section with long weakly restrained flanges.

Exercises:

1. Let the length of the vertical web be \( k(a) \). Rework the entire shear flow analysis for the C-channel including the check process by integrating the shear stress to obtain the statically equivalent shear forces.

2. ** Let the length of the top leg of the C-channel be \( a \), the length of the vertical web be \( 2a \), and the length of the bottom leg be \( 3a \). Rework the entire shear flow analysis for the C-channel including the check process by integrating the shear stress to obtain the statically equivalent shear forces.

3. ** Consider an angle iron (Figure 11.8) with horizontal leg length \( a \) and vertical leg length \( k(a) \). Locate the point(s) along the leg(s) where (a) the normal stress is zero, (b) the shear stress is zero.
WELDED MEMBERS

There are just two basic types of fillet welds: longitudinal and transverse. These welds appear in Figure 11.15. Both are generally assumed to resist shear forces, i.e., the welds are said to loaded in longitudinal shear and transverse shear respectively. However, as illustrated in Figure 11.16, it is evident that transverse welds experience a combination of normal and shear stresses, whereas longitudinal welds experience primarily shear stresses. (Moreover, these stresses vary along the length of the welds and are generally substantially larger at the ends than at the center.) The complexity of the actual stress state is circumvented by analyses employing elementary calculations coupled with allowable stresses given by appropriate Codes.

Elementary Analysis:

If we assume that the angle iron in Figure 11.15 is rigid-like, then the shear deformation (strain) of each weld element is the same throughout the basal plane. Accordingly, we assume for a Hookean material that the shear stress is uniform everywhere on the basal plane. Then, for any fillet weld, regardless of its orientation to the applied load,

\[ S_{ns} = \frac{F_{ns}}{\sum A_i} = \frac{F_{ns}}{A_{total}} \]

in which the total area refers to the basal plane.

The corresponding normal stress (caused by a force acting normal to the basal plane) is computed using the expression
Figure 11.15 Longitudinal and transverse fillet welds.
Figure 11.16  Longitudinal and transverse shear. Undeformed welds shown by dashed lines, deformed welds by solid lines.

Note: Longitudinal welds deform much more prior to fracture than do transverse welds. On the other hand, transverse welds are considerably stronger (about 40 to 45%) than longitudinal welds. Nevertheless, the allowable stress given in most Codes is the same regardless of weld orientation.
\[ S_{nn} = \frac{F_{nn}}{\sum A_i} = \frac{F_{nn}}{A_{total}} \]

It is assumed in this analysis that \( F_{nn} \) acts through the centroid of the weld areas in the basal plane.

Exercise

Compute the average shear stress \( S_{ns} \) and the average normal stress \( S_{nn} \) pertaining to the basal plane of the welded brackets in Sketches One and Two. Start your analysis by stating the statically equivalent force and moment acting at the centroid of the weld areas in the basal plane. Let \( P = 1500 \) Newtons and \( \theta = 50 \) degrees unless stated otherwise.

(a) Sketch One
(b) Sketch Two, 3mm fillet welds
(c) Sketch Two, 6mm fillet welds
28mm x 28mm x 3mm Angle Iron
BOLTED AND RIVETED MEMBERS

It is now common practice to use bolts to resist shear forces. (It used to be a design axiom that bolts are not used to resist shear forces. Bolts were meant to resist tensile forces and dowel pins were used to resist shear forces.)

If we assume that the member is rigid-like and that every bolt or rivet is deformed the same amount as the member tends to translate along the basal plane, then the transverse shear stress may be estimated using the elementary expression

\[ S_{ns} = \frac{F_{ns}}{\Sigma A_i} = \frac{F_{ns}}{A_{total}} \]

in which the total area refers to the cross sectional areas of the bolts or rivets.

The corresponding normal stress (associated with the force component of the statically equivalent load that acts normal to the basal plane) is computed using the expression

\[ S_{nn} = \frac{F_{nn}}{\Sigma A_i} = \frac{F_{nn}}{A_{total}} \]

where \( \vec{F}_{nn} \) acts through the centroid of the bolt or rivet pattern in the basal plane.

The assumption that each bolt or rivet resists a shear force is perhaps most credible when it is also
assumed that each bolt or rivet has yielded (and that the bolt or rivet material may be modeled as being elastic, perfectly plastic). This limiting case provides an upper bound (in theory) on the applied loads. Although this limit design procedure is useful in certain applications, it should not be used when either (a) the bolt experiences a tensile stress in addition to the transverse shear stress, or (b) the material does not exhibit an extraordinary amount of plastic deformation prior to fracture. The assumption that each bolt actually resists the computed shear force also appears credible for applications involving press-fit high strength bolts. Otherwise, alternative analyses based on alternative assumptions should be generated and evaluated.

Exercises:

Compute the average shear stress \( S_{ns} \) and the average tensile stress \( S_{nn} \) pertaining to the basal plane of the brackets in Sketches Three and Four. Start your analysis by stating the statically equivalent force and moment acting at the centroid of the bolt or rivet pattern in the basal plane. Let \( P = 1500 \) Newtons and \( \theta = 30 \) degrees unless stated otherwise.

(a) Sketch Three
(b) Sketch Four