CHAPTER NINE

STRESS ANALYSIS FOR THE AXIAL FORCE COMPONENT
OF INTERNAL LOADS
Introduction

This Chapter presents the stress analysis relevant to axial force components of internal loads. In addition, the concepts of stress distributions and maximum local stresses are introduced and discussed.

In particular, we illustrate in detail the solution to the state of stress in a tension test specimen, and then we extend this solution to more general applications, each time examining the stress distribution and searching for the maximum local shear stress. Subsequently, (ignoring certain complications) we can design against local yielding.
THE TENSION TEST: A DETAILED ANALYSIS

When we introduced the generalized Hooke's law relationships in Chapter 8, it was implicitly assumed that the one-dimensional and the three-dimensional models are compatible. Specifically, if we compare the expressions

\[ e_{nn} = \frac{S_{nn}}{E} \]

and

\[ e_{zz} = \frac{1}{E} [S_{zz} - \nu(S_{xx} + S_{yy})] \]

it is clear that we require the sum \( S_{xx} + S_{yy} \) to equal zero to compute \( E \) as the slope of the stress-strain diagram for \( S_{zz} \) plotted versus \( e_{zz} \). Since the normal stress is zero on the lateral surfaces of a tension specimen and the stress-strain relationship is valid throughout the entire gauge length portion of the specimen, it appears that the intuitive solution is \( S_{xx} = S_{yy} = 0 \). But, is this solution reasonable, viz., we know that the transverse strains \( e_{xx} \) and \( e_{yy} \) are nonzero. Accordingly, we might pose the following questions:

1. What are the distributions for \( S_{xx} \) and \( S_{yy} \), viz., how do these stresses vary from point to point in and along the gauge length of the tension specimen.

2. Is it possible to have stresses without strains. Is it possible to have stresses without strains.

We now perform a detailed analysis of the state of stress in and along the gauge length portion of a round tension specimen in the expectation of obtaining results that may help answer these questions.
STEP ONE -- EXAMINE THE OVERALL PROBLEM, ESTABLISH THE COORDINATE SYSTEM, AND STATE THE "KNOWN" (AND ASSUMPTIONS) AND THE Unknowns IN ANALYSIS

The logical choice of a coordinate system for a round (cylindrical) tension specimen is polar coordinates: R,θ,Z. Moreover, if we assume that (a) the material is homogeneous, viz., that the deformation of each infinitesimal element is the same regardless of its location in the gauge length portion of the tension specimen, and (b) the circular cross section is uniform along the entire gauge length, then we may argue that there is no θ nor Z variation in strain and that no shear deformations occur on R,θ,Z coordinate planes. Accordingly,

\[ e_{rr} = 0 = e_{\theta\theta} = e_{\theta Z} \]

and further assuming a (isotropic) Hookean material

\[ S_{rr} = G e_{rr} = 0 = S_{\theta\theta} = S_{\theta Z} \]

Consequently \( e_{rr}, e_{\theta\theta}, e_{zz} \) are principal strains and \( S_{rr}, S_{\theta\theta}, S_{zz} \) are principal stresses.

Based on the assumptions thus far we assert that \( e_{zz} \) is constant throughout the entire gauge length portion of the tension specimen. Using the argument that stress is proportional to strain, we can show that \( S_{zz} \) is constant across the entire cross section of the tension specimen everywhere along its gauge length. In turn, static equilibrium analysis indicates that

\[ S_{zz} = \frac{P}{A} \]

But, if we write the generalized Hooke's law relationship between \( S_{zz} \) and \( e_{zz} \)
9.3

\[ e_{zz} = \frac{1}{E} [S_{zz} - \nu(S_{rr} + S_{ee})] \]

then

\[ e_{zz} = \frac{1}{E} S_{zz} = \frac{1}{E} \frac{P}{A} \]

only if \((S_{rr} + S_{ee})\) equals zero at every location throughout the entire gauge length portion of the tension specimen.

Unknowns:

The unknowns are stresses \(S_{rr}\) and \(S_{ee}\). At present we can say definitely only that both stresses are finite and that

\[ S_{rr} \bigg|_{r=r_0} = 0 \]

in which \(r_0\) infers the outside radius of the specimen.

Problem:

Solve for \(S_{rr}\) and \(S_{ee}\) throughout the entire gauge length portion of the tension specimen.
STEP TWO -- WRITE THE DIFFERENTIAL EQUATIONS OF EQUILIBRIUM
FOR A TYPICAL INFINITESIMAL ELEMENT OF MATERIAL

Ordinarily this step in elasticity analysis merely involving writing the general form of the differential equations of equilibrium for the coordinate system of interest and then simplifying these equations in accordance with the assumptions stated in Step One. However, we shall take a different tack herein and present the (simplified) differential equations pertaining to the particular case of interest, viz., polar coordinates with no \( \theta \) variation (and no body forces).

Consider Figure 9.1. The typical infinitesimal element in (b) experiences only principal stresses. Static equilibrium, (c), requires that

\[
(\Sigma \vec{F}) \cdot \hat{n} = S'_{rr} (r+dr)dz - S_{rr} r dz - 2S_{ee} dr dz (\frac{dz}{dz}) = 0
\]

and, by inspection

\[
(\Sigma \vec{F}) \cdot \hat{e} = 0 \quad (\Sigma \vec{M}) \cdot \hat{\theta} = 0
\]

Substituting for \( S'_{rr} \) in the first scalar equation above gives

\[
\Sigma (\vec{F} \cdot \hat{n}) = 0 = [S_{rr} + \frac{\partial S_{rr}}{\partial r} (r+dr)dz] (r+dr)dz - S_{rr} (r dz) - S_{ee} dr dz d\theta
\]
Figure 9.1 Equilibrium Analysis for Typical Infinitesimal Element

\[ S'_{rr} = S_{rr} + \frac{\partial S_{rr}}{\partial r} dr \]
\[ S'_{ee} = S_{ee} \quad \text{(No } \theta \text{ Variation)} \]
\[ (\vec{F}_e \cdot \hat{n}) = S'_{rr} (r+dr) \, d\theta \, dr \, dz \]
\[ (\vec{F}_e \cdot \hat{r}) = -S'_{rr} r \, d\theta \, dr \, dz \]
\[ (\vec{F}_e \cdot \hat{\theta}) = -S_{ee} \, dr \, d\theta \, dz \left( \frac{de}{2} \right) \]
\[ (\vec{F}_e \cdot \hat{\phi}) = S_{ee} \, dr \, d\theta \, dz \]
\[ (\vec{F}_e \cdot \hat{\phi}) = -S_{ee} \, dr \, d\theta \, dz \left( \frac{de}{2} \right) \]
\[ (\vec{F}_e \cdot \hat{\phi}) = -S_{ee} \, dr \, d\theta \, dz \]

\[ \vec{F}_e = [S'_{ee} \, dr \, d\theta \, dz] (\hat{\phi}_+) \]

\[ \vec{F}_e = [S_{ee} \, dr \, d\theta \, dz] (\hat{\phi}_-) \]

\[ e_+ = e + \frac{de}{2} \]
\[ e_- = e - \frac{de}{2} \]

Note: For small angles: \[ \sin(e) \approx \frac{de}{2} \]
\[ \cos(e) \approx 1 \]
which then simplifies to give the result

\[ S_{rr} - S_{ee} + r\left[ \frac{\partial S_{rr}}{\partial r} \right] + dr \left[ -\frac{\partial S_{rr}}{\partial r} \right] = 0 \]

But the last term is negligible as \( dr \) approaches zero in the limit. Hence, static equilibrium is satisfied when

\[ \frac{\partial S_{rr}}{\partial r} + \frac{S_{rr} - S_{ee}}{r} = 0 \quad \text{[A]} \]

Since only one variable is involved the partial derivative may be replaced by the ordinary derivative.

Exercises:

1. Consider the element in Sketch A below. State expressions for the stresses acting on the outboard faces, allowing both \( \Theta \) and \( R \) variation.

2. Given Sketch B, show that static equilibrium requires that

\[ \frac{\partial S_{rr}}{\partial r} + \frac{1}{r} \left[ \frac{\partial S_{re}}{\partial \Theta} \right] + \frac{S_{rr} - S_{ee}}{r} \cdot B_r = 0 \]

\[ \frac{1}{r} \left[ \frac{\partial S_{ee}}{\partial \Theta} \right] + \frac{2S_{re}}{r} + \frac{S_{re}}{r} + B_{\Theta} = 0 \]

\[ S_{re} = S_{er} \]
Sketch A
Sketch B
where $B_r$ and $B_\theta$ are body forces (e.g., due to acceleration). What units are appropriate for these body forces.

3. Suppose the infinitesimal element were rotating with a constant angular velocity. Would $B_r$ equal zero. Would $B_\theta$ equal zero. Suppose the infinitesimal element were rotating with an increasing angular velocity. Would $B_r$ equal zero. Would $B_\theta$ equal zero.

4. If $B_\theta$ equals zero, show that for no $\theta$ variation the second differential equation in Exercise 2 is satisfied only when $S_{r\theta} = 0$ or $S_{r\theta} = C/r^2$. Is the latter solution acceptable. Reconsider the problem for the case where $B_\theta \neq 0$. (Physically describe the case where $B_\theta \neq 0$.)
STEP THREE -- WRITE THE STRAIN-DISPLACEMENT RELATIONSHIPS
FOR A TYPICAL INFINITESIMAL ELEMENT OF MATERIAL

Consider Figure 9.2. For no θ variation the angle δθ is invariant under deformation (because the total angle 2π radians cannot change for a round specimen). This means the radial element sides can only be displaced in the radial direction under deformation.

The normal strain in the radial direction is thus

\[ e_{rr} = \frac{\left( r_o - r_i \right)_{\text{deformed}} - \left( r_o - r_i \right)_{\text{undeformed}}}{\left( r_o - r_i \right)_{\text{undeformed}}} \]

\[ = \frac{dr + \left( \frac{3u}{r} \right)dr - dr}{dr} \]

\[ e_{rr} = \frac{3u}{r} \quad \text{[B.1]} \]

The corresponding normal strain in the circumferential direction is

\[ e_{θθ} = \frac{r_{ave,\text{deformed}} - r_{ave,\text{undeformed}}}{r_{ave,\text{undeformed}}} \]

\[ = \frac{r + u + \frac{dr}{2} + \frac{1}{2} \left( \frac{3u}{2r} \right)dr - r - \frac{dr}{2}}{r + \frac{dr}{2}} \]

But,

\[ e_{θθ} = \frac{u}{r} \quad \text{[B.2]} \]

when \( e_{θθ} \) is evaluated as \( dr \) approaches zero in the limit.
Figure 9.2 Displacement relationships for the typical infinitesimal element.

Deformed Element (Dashed Line)

Undeformed Element (Solid Line)

Note: The value of the displacement is \( u \) at radius \( r \), but its value is \( u + (\partial u/\partial r)dr \) at radius \( (r + dr) \). (This difference in displacement causes deformation.)

Dimensions Before and After Deformation:

\[
\begin{align*}
    r_{i, \text{undeformed}} &= r \\
    r_{o, \text{undeformed}} &= r + dr \\
    r_{ave, \text{undeformed}} &= r + \frac{dr}{2} \\
    r_{i, \text{deformed}} &= r + u \\
    r_{o, \text{deformed}} &= r + dr + u + \frac{\partial u}{\partial r}dr \\
    r_{ave, \text{deformed}} &= r + \frac{dr}{2} + u + \frac{1}{2} \frac{\partial u}{\partial r}dr
\end{align*}
\]
STEP FOUR -- WRITE THE STRESS-STRAIN RELATIONSHIPS FOR
THE TYPICAL INFINITESIMAL ELEMENT OF MATERIAL

We assume herein that the generalized Hooke's law
expressions describe the stress-strain relationships for the given
material (which is linear, elastic, homogeneous and isotropic). Then,

\[ e_{rr} = \frac{1}{E} [S_{rr} - \nu (S_{ee} + S_{zz})] \]
\[ e_{ee} = \frac{1}{E} [S_{ee} - \nu (S_{zz} + S_{rr})] \]
\[ e_{zz} = \frac{1}{E} [S_{zz} - \nu (S_{rr} + S_{ee})] \]

\[ e_{re} = \frac{1}{G} S_{re} \]
\[ e_{ez} = \frac{1}{G} S_{ez} \]
\[ e_{zr} = \frac{1}{G} S_{zr} \]
STEP FIVE — RE-EXPRESS THE DIFFERENTIAL EQUATIONS OF EQUILIBRIUM IN TERMS OF DISPLACEMENTS, USING THE STRESS-STRAIN RELATIONSHIPS

Our objective now is to satisfy Equations [A],[B], and [C] simultaneously. This is conveniently accomplished by re-expressing the differential equation of equilibrium in terms of the displacement \( u \) using the strain definitions in conjunction with the generalized Hooke's law expressions. The associated manipulations are outlined below.

First, substituting the definitions for \( e_{rr} \) and \( e_{ee} \) into the generalized Hooke's law expressions gives

\[
\frac{\partial u}{\partial r} = \frac{1}{E} [S_{rr} - u(S_{ee} + \frac{P}{A})]
\]

and

\[
\frac{u}{r} = \frac{1}{E} [S_{ee} - u(S_{rr} + \frac{P}{A})]
\]

Then, viewing these two expressions as two equations in two unknowns, \( S_{rr} \) and \( S_{ee} \), we may write

\[
\left[ \frac{1}{E} \right] S_{rr} + \left[ - \frac{v}{E} \right] S_{ee} = \left( \frac{\partial u}{\partial r} + \frac{uP}{EA} \right)
\]

\[
\left[ - \frac{v}{E} \right] S_{rr} + \left[ \frac{1}{E} \right] S_{ee} = \left( \frac{u}{r} + \frac{uP}{EA} \right)
\]

In turn, solving for \( S_{rr} \) and \( S_{ee} \) gives

\[
S_{rr} = \left( \frac{\partial u}{\partial r} + \frac{uP}{EA} + \frac{uv}{r} + \frac{u^2P}{EA} \right) / \left( \frac{1}{E} \right) (1 - u^2)
\]
and

\[ S_{ee} = (u \frac{\partial^2 u}{\partial r^2} + \frac{v^2 P}{EA} + \frac{u}{r} + \frac{vP}{EA})/(\frac{1}{E})(1-v^2) \]

Finally we substitute these expressions into the differential equations of equilibrium and simplify to obtain

\[ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} = 0 \quad [D] \]

Exercise:

Check the details of the manipulations that lead to the differential equation [D] above. Is it correct.
STEP SIX -- SOLVE THE SUMMARY EQUATIONS IN GENERAL FORM

We shall now solve the differential equation [D]

\[ \frac{d^2u}{dr^2} + \frac{1}{r} \frac{du}{dr} - \frac{u}{r^2} = 0 \]

First, we re-express this differential equation in the form

\[ \frac{d^2u}{dr^2} + \frac{d}{dr}[\frac{u}{r}] = 0 = \frac{d}{dr}[\frac{du}{dr} + \frac{u}{r}] \]

and then we integrate to obtain

\[ \frac{du}{dr} + \frac{u}{r} = C_1 \]

But this new differential equation may in turn be re-expressed as

\[ \frac{1}{r} \frac{d}{dr}[ur] = C_1 \quad \text{or} \quad \frac{d}{dr}[ur] = C_1 r \]

Hence, subsequent integration gives

\[ ur = \frac{C_1 r^2}{2} + C_2 \]
and solving for displacement \( u \), we obtain

\[
  u = \frac{C_1}{2} r + \frac{C_2}{r}
\]

This result may now be used to state the strains \( e_{rr} \) and \( e_{\theta\theta} \) in terms of \( r \), and these strains in turn may be used in conjunction with the generalized Hooke's law expressions to express \( S_{rr} \) and \( S_{ee} \) as a function of \( r \), viz.,

\[
  S_{rr} = A - \frac{B}{r^2}
\]

and

\[
  S_{ee} = A + \frac{B}{r^2}
\]

in which

\[
  A = \frac{E}{(1-v^2)} \left[ \frac{C_1}{2} (1+u) + \frac{uP}{EA} (1+u) \right]
\]

and

\[
  B = \frac{E}{(1-v^2)} [C_2 (1-u)]
\]

Exercise:

Verify the expressions given above for \( A \) and for \( B \).
STEP SEVEN -- SATISFY THE BOUNDARY CONDITIONS

In Step One we noted that $S_{rr}$ and $S_{ee}$ are finite and that

$$S_{rr} \big|_{r=r_0} = 0$$

From the latter condition we see that

$$S_{rr} = A - \frac{B}{r_0^2} = 0$$

Hence,

$$A = \frac{B}{r_0^2}$$

Substituting this result gives

$$S_{rr} = B\left[\frac{1}{r_0^2} - \frac{1}{r^2}\right]$$

But $S_{rr}$ must be finite when $r$ equals zero, which dictates that $B = 0$. Hence, $A = B = 0$, and consequently

$$S_{rr} = S_{ee} = 0$$

everywhere in and along the entire gauge length portion of the tension specimen.
STEP EIGHT -- SUMMARIZE AND PLOT THE ANALYTICAL RESULTS

The analytical results in this tension test example are so elementary that a summary is superfluous. Nevertheless we should point out that \( C_1 / 2 = -\frac{uP}{EA} \) and \( C_2 = 0 \) for the boundary conditions given; hence,

\[
u = -\frac{uPr}{EA}
\]

and

\[
er = \frac{du}{dr} = \frac{uP}{EA} = -\nu e_{zz}
\]

\[
e_\theta = \frac{u}{r} = -\frac{uP}{EA} = -\nu e_{zz}
\]

Both \( e_{rr} \) and \( e_{ee} \) are constant throughout the gauge length portion of the tension specimen.

(It is, we trust, clear to the discerning student that once we get outside of the gauge length portion of the tension specimen, the strain changes from point to point and thus the corresponding analytical solution is much more complex. Moreover, it is well known that \( S_{zz} \) at the fillets of the tension specimen is larger than \( S_{zz} \) throughout the gauge length portion of the tension specimen.)
STEP NINE -- DISCUSS AND EVALUATE THE ANALYTICAL RESULTS

We now face the issue: How can a material deform in a direction in which no stress acts.

This question cannot be answered from an engineering mechanics perspective because of the distinction between the actual and theoretical stress-strain relationships. The behavior of the material in the tension specimen is governed by the actual stress-strain relationships which are unknown and can only be deduced by appropriate experimentation and measurement. On the other hand, the generalized Hooke's law expressions are theoretical, having been constructed to describe the actual stress-strain phenomena exhibited by the specific class or group of materials that may accurately be modeled as being linear, elastic, homogeneous and isotropic.

The following elementary materials science analysis may provide some insight regarding the nature of the transverse contraction of a hypothetical material under tensile loading (but it also demonstrates our lack of knowledge of fundamental phenomenon). Consider Figure 9.3. If we assume the hypothetical material is composed of atoms in a close-packed arrangement with the special orientation shown in (a), then an estimate of Poisson's ratio is possible considering only the four atoms associated with the parallelogram in (b). Assume that atoms A and C separate, while the distance (bond length 2r) remains constant between A-B, B-C, C-D, and D-A. The centers of atoms B and D thus approach one another. For the hypothetical geometry assumed, Poisson's ratio is 0.33. This value agrees reasonably well with measured values for most materials that may be viewed as being isotropic, but v may be as high as 0.5 for rubber and as low as 0.0 for cork. (Values of v well in excess of 0.5 are common in a specific direction for anisotropic materials undergoing plastic deformation.)
Figure 9.3  Poisson's ratio analysis for a hypothetical material. For the close-packed plane loaded in the \( Z \) direction, (a):

\[
e_{zz} = \frac{\delta_z}{r}
\]

\[
e_{yy} = -\frac{\delta_y}{\sqrt{3} \ r}
\]

But, from (c), \( \delta_y = \delta_z / \sqrt{3} \)

Thus, \( e_{yy} / e_{zz} = -1/3 \).

Note:
\( \delta_y = \delta_z / \sqrt{3} \)
for constant bond length 2\( r \)
STEP TEN -- EXTEND AND APPLY THE ANALYTICAL RESULTS

We now turn to gleaning benefits from our analytical efforts.

APPLICATIONS

1. Shrink Fit

Consider the state of stress resulting when a hub is shrunk on a round shaft. If we assume that the hub geometry is such that associated deformation of the shaft exhibits no \( \theta \) variation, and if we further restrict our analysis to the portion of the shaft remote to the edges of the hub so that the assumption of no \( \theta \) variation is credible, then the general solution pertains and we need merely set \( P \) equal to zero and select constants \( A \) and \( B \) such that the appropriate boundary conditions are satisfied. For finite stress at \( r = 0 \) and

\[
S_{rr}\bigg|_{r=r_0} = -P_0 \quad (P_0 = \text{Pressure at } r_0)
\]

\( B = 0 \) and \( A = -P_0 \). Hence,

\[
S_{rr} = S_{rr} = -P_0
\]

everywhere, and for completeness we also note that

\[
S_{zz} = S_{re} = S_{ez} = S_{rr} = 0
\]

everywhere (viz., throughout the portion of the round shaft where the assumptions made in analysis appear credible).
Exercise:

If you had to estimate the interference pressure \( P_0 \), list geometrical and materials considerations that are important.

2. Thick-Walled Cylinder

Consider the thick-walled cylinder schematic in Figure 9.4 and the associated typical infinitesimal stress element. Let the external pressure acting on the outside surface of the cylinder be denoted \( P_0 \) and the internal pressure acting on the inside surface be denoted \( P_i \). The appropriate boundary conditions are

\[
S_{rr}|_{r=r_i} = -P_i
\]

and

\[
S_{rr}|_{r=r_o} = -P_o
\]

These two boundary conditions are satisfied when (simultaneously)

\[
A - \frac{B}{r_i^2} = -P_i
\]

and

\[
A - \frac{B}{r_o^2} = -P_o
\]

Hence,

\[
A = \frac{P_i r_i^2 - P_o r_o^2}{(r_o^2 - r_i^2)}
\]

and

\[
B = \frac{(P_i - P_o) r_i^2 r_o^2}{(r_o^2 - r_i^2)}
\]
Thick-walled cylinder schematic with a typical infinitesimal stress element displayed. $S_{rr}$ = radial stress, $S_{zz}$ = circumferential (or hoop) stress, and $S_{zz}$ = longitudinal stress.

Figure 9.4
Accordingly,

\[ S_{rr} = \frac{P_1 r_1^2 - P_0 r_0^2}{(r_0^2 - r_1^2)} - \left[ \frac{(P_1 - P_0) r_1^2 r_0^2}{(r_0^2 - r_1^2)} \right] \left( \frac{1}{r^2} \right) \]

[9.1]

and

\[ S_{ee} = \frac{P_1 r_1^2 - P_0 r_0^2}{(r_0^2 - r_1^2)} + \left[ \frac{(P_1 - P_0) r_1^2 r_0^2}{(r_0^2 - r_1^2)} \right] \left( \frac{1}{r^2} \right) \]

Exercise:

Verify the expressions given for A, B, S_{rr} and S_{ee}.

Figures 9.5(a), (b), and (c) display the radial stress distributions associated with only internal, only external, and both internal and external pressures imposed on a thick-walled cylinder with \( r_1 = 225 \text{mm} \) and \( r_0 = 400 \text{mm} \). In these examples, \( P_0 = 400 \text{ Mpa} \) and \( P_1 = 200 \text{ Mpa} \). Observe that in all cases \( S_{ee} \) takes on its largest absolute value at the inside radius.

Figures 9.5(a), (b), and (c) also display the radial distributions of the maximum shear stress \( S_{r/e} \). Recall that the Mohr's circle analysis for a tension test specimen indicated that the maximum shear stress occurs on a plane oriented at 45 degrees to the planes of principal stress, e.g., on the plane indicated in Figure 9.6. Observe that in all cases the maximum shear stress \( S_{r/e} \) takes on its largest value at the inside surface of the cylinder (where \( S_{r/e} = S_{r/e, \text{max}} \)).
Radial distribution of principal stresses $S_{rr}$ and $S_{ee}$ and the corresponding maximum shear stress $S_{r/e}$.

Figure 9.5(a)
Radial distribution of principal stresses $S_{rr}$ and $S_{\theta \theta}$ and the corresponding maximum shear stress $S_{r/\theta}$.

Figure 9.5(b)
Radial distribution of principal stresses $S_{rr}$ and $S_{ee}$ and the corresponding maximum shear stress $S_{r/e}$.

Figure 9.5(c)
Given principal stresses $S_{rr}$ and $S_{ee}$, the associated maximum shear stress occurs on a 45 degree plane and has magnitude

$$S_{r/e} = \frac{\text{Abs. Value} [S_{rr} - S_{ee}]}{2}$$

(the radius of the Mohr's circle. It is important not to confuse $S_{r/e}$ with the coordinate shear stress $S_{re}$ pertaining to the $R, \theta$ axes.)
Exercises:

1. Verify the stress distribution plots for the thick-walled cylinder example. Add the vertical stress scale used to construct the plot in Figure 9.5: (a), (b), or (c).

2. Show that the maximum shear stress $S_{r/e, \text{max}}$ is given by

$$S_{r/e, \text{max}} = \frac{(P - P_0) r^2}{r^2_0 - r^2_1}$$

Now that we have discussed the radial distribution of the maximum shear stress $S_{r/e}$, it may appear tempting to consider the question of what pressure would cause the thick-walled cylinder to yield. However, in terms of design perspective it is much more meaningful to reconsider the thick-walled cylinder schematic in Figure 9.4 and to point out that the analysis ignores the ends of the cylinder and permits no inlet or outlet ports in the cylinder wall (or the associated pads and bosses). Thus our analysis is not yet sufficiently complete to consider the practical aspects of yielding.

Our objective at present is to look at various stress distributions and to learn to search for the maximum local shear stress. Eventually we will consider the problem of local yielding, but our analysis will be very elementary and rather limited in application. This limitation only serves to emphasize the critical perspective that elementary analyses must be enhanced by experimental stress analyses and advanced theory (e.g., finite element analyses) to have reliable design application.
2a. Shrink fit between two thick-walled cylinders

The state of stress resulting from a shrink fit between two thick-walled cylinders is of both academic and practical interest. First, we note that the boundary conditions relevant to establishing the values of four constants (A and B for each cylinder) must be elucidated. Clearly, two boundary conditions pertain to the pressures acting at the inside and outside surfaces of the assembly. Another boundary condition is that the respective pressures at the interface between the two cylinders are equal in magnitude. The final boundary condition is left to the student to discern. (It is based on a physical understanding of what happens as the shrink fit takes place.)

Figure 9.7 shows the radial distribution of the circumferential stress in both cylinders. Detailed analysis will show that the prestressed assembly will withstand a higher internal pressure without yielding than if the assembly were replaced by a single cylinder with the same overall dimensions.

3. Bending of Curved Members

Figure 9.8 shows a segment of a homogeneous ring with a rectangular cross section. The segment is subjected to pure bending as illustrated. Advanced analysis shows that (for locations remote to the ends of the segment)

\[ S_{rr} = \frac{4M}{br_0K} \left[ (1 - \frac{r_1^2}{r_2^2}) \log_e \left( \frac{r}{r_1} \right) - \left( 1 - \frac{r_1^2}{r_2^2} \right) \log_e \left( \frac{r_2}{r_1} \right) \right] \]

and

\[ S_{ee} = \frac{4M}{br_0^2K} \left[ (1 - \frac{r_1^2}{r_2^2}) (1 + \log_e \left( \frac{r}{r_1} \right)) - \left( 1 + \frac{r_1^2}{r_2^2} \right) \log_e \left( \frac{r_2}{r_1} \right) \right] \]
Figure 9.7  Circumferential (hoop) stresses for a shrink fit between two thick-walled cylinders. Subsequent pressurization of the inner cylinder causes the compressive circumferential stress to first decrease to zero before becoming tensile.
Figure 9.8 Segment of a ring with uniform thickness and width, subjected to pure bending. The analysis pertains to all portions of the ring remote to the bending moments $\vec{M}$. 
where

\[ K = \left[ \left( 1 - \frac{r_i^2}{r_o^2} \right)^2 - 4 \frac{r_i^2}{r_o^2} \log_e \left( \frac{r_o}{r_i} \right) \right] \]

and \( b \) equals the width of the ring segment, Figure 9.8.

Exercises:

1. Let \( M = 200 \text{ Newton-meters} \), \( b = 24 \text{ mm} \), \( r_i = 60 \text{ mm} \), and \( r_o = 90 \text{ mm} \). Plot \( S_{rr} \) and \( S_{ee} \).

2. Examine the analytical expressions for \( S_{rr} \) and \( S_{ee} \). Which stress is larger. Where is this stress largest?

3. Using a Mohr's circle analysis, what is the value of \( S_{r/e, \text{max}} \). Where does it occur.

4. Rotating Ring

Figure 9.9 displays a typical infinitesimal stress element located in a homogeneous ring with constant width \( b \). The ring is rotating with a constant angular velocity \( \omega \). Advanced analysis shows that

\[ S_{rr} = \frac{3 + \nu}{8} \left[ \frac{r_i^2}{r_o^2} + 1 - \frac{r_i^2}{r_o^2} - \frac{r_i^2}{r_o^2} \rho r_o^2 \omega^2 \right] \]

and
Figure 9.9  Homogeneous ring with uniform thickness and width, density $\rho$, rotating with constant angular velocity $\omega$.

Body Force $B_r = [w^2r][\rho dV] = [w^2r][\rho r dr dz]$
\[ S_{ee} = \frac{3+b}{8} \left[ \frac{r_1^2}{r_0^2} + 1 - \left( \frac{1+3b}{2} + b \right) \frac{r_2^2}{r_0^2} + \frac{r_1^2}{r_2^2} \right] \rho r_0^2 \omega^2 \]

Exercises:

1. Assume a forged 4340 steel disc with \( r_1 = 100 \text{mm} \) and \( r_0 = 350 \text{mm} \) rotating at 3600 RPM. Plot \( S_{rr} \) and \( S_{ee} \).

2. Examine the analytical expressions for \( S_{rr} \) and \( S_{ee} \). Which stress is larger. Where is this stress largest.

3. From a Mohr's circle analysis, what is the largest shear stress. Where does it occur.
STRESS CONCENTRATION

The concept of stress concentration is derived from an understanding of stress distributions. Suppose, for example, that we examine the stress distribution around a very small hole in a very large plate that experiences a uniaxial tensile (nominal) stress in regions remote to the hole, Figure 9.10. This stress distribution is given by the Kirsch solution

\[
S_{rr} = \frac{S_{nn,\text{nom}}}{2} \left[ (1 - \frac{a^2}{r^2}) + (-1 + \frac{a^2}{r^2} - \frac{3a^4}{r^4})\cos2\theta \right]
\]

\[
S_{\theta\theta} = \frac{S_{nn,\text{nom}}}{2} \left[ (1 + \frac{a^2}{r^2}) + (1 + \frac{3a^4}{r^4})\cos2\theta \right]
\]

\[
S_{r\theta} = \frac{S_{nn,\text{nom}}}{2} (1 + 2\frac{a^2}{r^2} - \frac{3a^4}{r^4})\sin2\theta
\]

in which \(a\) is the radius of the hole, and \(S_{rr}, S_{\theta\theta}, S_{r\theta}\) are the coordinates stresses at location \([r, \theta]\). The location and magnitude of the maximum shear stress, \(S_{n/t,\text{max}}\) are established as follows. First, at each location we determine the principal stress state

\[
\begin{bmatrix}
S_{rr} & S_{r\theta} & 0 \\
S_{r\theta} & S_{\theta\theta} & 0 \\
0 & 0 & 0
\end{bmatrix} \leftrightarrow \begin{bmatrix}
S_{11} & 0 & 0 \\
0 & S_{22} & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

and then we search with regard to location to find the maximum shear stress associated with the principal state of stress. For this elementary example, the maximum shear stress occurs at \(r = a\) and \(2\theta = 0\) degrees. Here
Figure 9.10  

\( S_{ee} \) distribution acting on cross section through the hole and at right angles to the direction of the applied (uniform) nominal normal stress \( S_{nn,\text{nom}} \).

\[ S_{nn,\text{nom}} = \text{Nominal Normal Stress} \]

\[ S_{pp,\text{max}} = S_{ee} \bigg|_{r=a} = 3S_{nn,\text{nom}} \]

Conceptual plane of interest

Geometry: Small hole in an infinite (thin) plate which is loaded as shown.

Note: The stress gradient \( d(S_{ee})/dr \) is much steeper at the edge of the hole than sketched above. Stress gradients are always very steep in the immediate vicinity of the stress concentrator.
\[ S_{11} = S_{pp, \text{max}} = 3S_{nn, \text{nom}} \text{ and } S_{22} = 0. \text{ Hence, } S_{1/2, \text{max}} = \left(\frac{3}{2}\right) S_{nn, \text{nom}}. \]

Suppose we reconsider the notion of a maximum stress search and this time do not specify what maximum stress we seek. What single value conveys the most information. Clearly the maximum principal stress. This maximum principal stress information may be summarized in very succinct format by the expression

\[ S_{pp, \text{max}} = K_t S_{\text{nominal}} \quad [9.2] \]

in which \( K_t \), the (theoretical) stress concentration factor, is equal to three in Figure 9.10.

Stress Concentration Factors

There are two fundamental types of stress concentration factors of interest in design analysis:

1. Academic: stress concentration factors, \( K_{t, pp} \), for estimating the maximum principal stress \( S_{pp, \text{max}} \) in a member that experiences only an elementary internal loading, e.g., only pure bending.

2. Applied: stress concentration factors, say, \( K_{t, n/t} \) for estimating the maximum shear stress \( S_{n/t, \text{max}} \) in a member that experiences "combined" internal loads, e.g., bending, torsion, and an axial force component simultaneously.

With few exceptions the stress concentration factor information found in the literature, e.g., Figure 9.11, pertain to stress states associated with elementary modes of internal loading, e.g., only axial forces, only pure bending, only torsion.
Principal stress concentration factors for long slender straight members with circular cross sections, experiencing only an internal axial force component $F_{nn}$. $S_{nn, nom} = F_{nn}/A = F_{nn}/(wd^2/4)$.

(a) Grooves

(b) Fillets
This academic stress concentration information must somehow be used in design to establish local stress states relevant to (a) the "combination" of internal loads generated by the critical service load state, and (b) the assumed failure criterion, in our case, yielding according to the shear stress criterion. We shall discuss the details of estimating \( S_{n/t,\text{max}} \) using academic stress concentration factors in Chapter 13.

In Figure 9.11(a) our intuition indicates that the point at which the maximum principal stress occurs lies at the exact center of the groove. However, in (b) the point at which the maximum principal stress occurs is not intuitively obvious. This lack of information regarding the exact location of the maximum principal stress is typical for most stress concentrators, e.g., splines, keyways, threads, etc. However, the lack of information regarding the principal stress state \( (S_{11}, S_{22}, S_{33}) \) at (say) the location of the maximum principal stress represents a much more serious problem in design analysis. Often, the only way this problem can be overcome is to study the literature and read the publications in which the relevant principal stress concentration factor developmental work was originally presented. This effort is indeed worthwhile.

Design stress analysis would indeed be straightforward if for every stress concentrator encountered we could compute the nominal stress and find the appropriate stress concentration factor. In fact, this is seldom the case. Usually we need employ experimental stress analysis techniques to establish the location of interest and to determine the corresponding state of stress. The stress analysis procedure based on nominal stress calculations modified by use of stress concentration factors has primary
application in the preliminary stages of design when no better local stress information is available (e.g., before a prototype has been developed). But even if a given member or component geometry and loading are particularly amenable to this form of stress analysis, we should not lose sight of the fact that such calculations represent only a small part of the overall design stress analysis, which always includes extensive effort in the area of experimental stress analysis, and which should include as much finite element analysis (Figure 9.12) as practical.
Figure 9.12  Schematic illustration of the finite element representation of a hypothetical structural part. The solid part is viewed as being built of small triangular elements, pin-connected at their corners.
Exercise Set One:

Locate the point of maximum principal stress. Show that the stress concentration factor, $K_{t,pp}$, is equal to four.

Geometry: Very small hole in a very large (thin) plate which experiences the biaxial state of stress shown in regions remote to the hole (tensile in one direction, compressive in the orthogonal direction).
Exercise Set Two:

1. In Figure 9.11(a) sketch the infinitesimal element at the bottom of the groove. Show $S_{oo}$ on this element. Let $S_{\phi \phi}$ and $S_{rr}$ be the hoop and radial stresses respectively.

2. In Figure 9.11(b) sketch the infinitesimal element near the center of the fillet. Show $S_{oo}$ on this element. Let $S_{\phi \phi}$ and $S_{rr}$ be the hoop and radial stresses respectively.

3. In Figure 9.11(a) state the principal stress concentration factor when $D/d = 1.5$ and $r/d = 0.125$.

4. In Figure 9.11(b) state the principal stress concentration factor when $D/d = 1.5$ and $r/d = 0.125$.

5. Given the example internal load diagrams below, find the maximum principal stress associated with the compressive axial force component of the internal loads. Consider each groove to locate the critical cross section. Size the throat diameters of the other two grooves such that the maximum principal stress is the same at all three grooves.
Example Internal Load Diagrams
DESIGNING AGAINST LOCAL YIELDING

If in preliminary design, only data for the tensile yield strength, $S_{\text{nn},Y,\text{tension}}$, were known, we would design such that the maximum local shear stress is less than the shear stress associated with the onset of yielding in a tension test. For example, given a thick-walled cylinder subjected to internal pressure only, we would design such that (ignoring the stress concentrations at the inlet and outlet ports)

$$S_{r/e,\text{max nom}} = \frac{P_{r0} r^2}{r_0^2 - r_1^2} < \frac{1}{2} S_{\text{nn},Y,\text{tension}}$$

This design condition is illustrated in Figure 9.13. If the stress concentration factor for shear stress at the critical port were known, the design inequality would be modified as follows

$$K_{t,r/e} S_{r/e,\text{max nom}} = K_{t,r/e} \frac{P_{r0} r^2}{r_0^2 - r_1^2} < \frac{1}{2} S_{\text{nn},Y,\text{tension}}$$

Finally, the design inequality takes on a quantitative interpretation when we consider a factor of safety. At this point in our discussion we must unfortunately define the factor of safety rather naively. Nevertheless, it appears reasonable to restate the design inequality as

$$K_{t,r/e} S_{r/e,\text{max nom}} = K_{t,r/e} \frac{P_{r0} r^2}{r_0^2 - r_1^2} < \frac{1}{2} S_{\text{nn},Y,\text{tension}} / (FS)$$
Designing against local yielding.

Mohr's circle (b) below pertains to the thick-walled cylinder analysis of Figure 9.6(b) in which longitudinal stress $S_{zz}$ was arbitrarily ignored to simplify this introductory presentation. $S_{r/e} = S_{r/e,max}$ because $S_{r/e}$ is evaluated at the inside surface of the cylinder. ($S_{r/e,max} = S_{r/e,max}$ now because the stress concentration is ignored in the stress analysis.)

(a) Mohr's circle analysis for the stress in an infinitesimal element taken from a tension test specimen when yielding has just occurred.

Maximum allowable shear stress equals (say) $S_{nn,Y,tension}/2$

(b) Mohr's circle analysis for the stress in an infinitesimal element taken from the inner surface of a thick-walled cylinder, refer Figure 9.6(b).
in which FS is the naive factor of safety.

No definition for a design factor of safety can be viewed as being realistic without considering factors such as residual stress, nonuniform strength distributions, service load states and histories, environments, etc. These factors are considered in Chapter 14. (But no factor of safety is without criticism.)

Exercises:

1. Extend the exercise (page 9.36) pertaining to the stress distribution for a rotating disc. Assume that local yielding will occur at a bolt hole near the inside diameter where $K_{t,n/t} = 2.6$. If a naive factor of safety of 8 is required, what is the minimum allowable tensile yield strength for the 4340 steel. Is this value reasonable.

2. Extend the exercise (page 9.34) pertaining to the stress distribution in a bent ring segment. Suppose the outside surface has a notch for which $K_{t,n/t} = 1.8$. What are the minimum allowable values for tensile and compressive yield strength when a naive factor of safety of 6 is required. Is 1020 cold-rolled steel an acceptable material.
RESIDUAL STRESS

Suppose we now consider briefly what happens when the local shear stress in the immediate vicinity of a stress concentrator exceeds the shear stress associated with yielding in a tension test. Actually this is a very complex problem, but we shall ignore the complications here and deal strictly with an elastic, perfectly plastic material model and its associated idealized yielding behavior. Our purpose at present is to illustrate the basic ideas of residual stress for a sheet tension specimen with a central hole, Figure 9.14.

Consider Figure 9.14. Assume that the bar is 24mm wide, 6mm thick, with a 6mm hole drilled through its center, and is subjected to a nominal axial stress of 280 Mpa. From Figure 9.15 we see that the stress concentration factor is 2.44. Hence the (maximum principal) circumferential stress at the edge of the hole would be 683 Mpa if no yielding occurs. But, assuming the tensile yield strength is 500 Mpa, the maximum value that the circumferential stress can take on is 500 Mpa, sketch (b). When this bar is unloaded the material at the edge of the hole unloads elastically resulting in a compressive residual stress equal to (minus) 163.2 Mpa. If the bar were subsequently reloaded it could withstand a nominal stress of 280 Mpa before further local yielding would take place.

Exercise:

Calculate the value of $S_{ee}$ at the edge of the hole when further yielding is imminent upon reloading the bar in Figure 9.14, i.e. when $S_{nn, nom}$ again reaches 280 Mpa.
Figure 9.14  Residual stress distribution resulting from localized yielding at the edge of the hole.

(a) $S_{ee}$ stress distribution with no yielding. Compare to (b).

(b) $S_{ee}$ stress distribution with local yielding. Compare to (a).

(c) $S_{ee}$ residual stress distribution equals (b)-(a), viz. the distribution with yielding minus the distribution associated with elastic unloading.

Note: The maximum value of $S_{ee}$ at the edge of the hole associated with elastic unloading could be as high as $2S_{nn,Y,tension}$ before local yielding would also occur upon unloading (assuming an elastic perfectly plastic material).
Note: The hole must be centered in the bar, plate, or sheet relative to dimension h. The analytical development for the stress concentration assumes plane stress, viz., thin members where b is very small compared to h. The curves are only approximate for thicker members, generally about 5–10% low.

Figure 9.15  Principal stress concentration factor for long slender straight members with rectangular cross sections, experiencing only an internal axial force component $F_{nn}$. $S_{nn, nom} = F_{nn}/A = F_{nn}/(b(b-d))$. 
CONSTRAINED (LOCAL) YIELDING

Geometric compatibility dictates that the strains in a plastic region display continuity relative to the strains in immediately adjacent elastic regions. Thus, elastic regions constrain plastic deformations that would otherwise occur. In particular, the strain at the root of a sharp notch is approximately the same whether local yielding occurs or not — because the stress gradient is so large that only a very small amount of material actually yields (in the sense that X-ray measurements indicate a slight drop in the elastic distortion of the lattice of polycrystalline metals). But, unless massive plastic flow takes place at the lateral surfaces (where essentially a plane stress state prevails), this local yielding is undetectable by conventional measuring techniques. For example, Figure 9.16 shows the strain at the inside surface of a central hole in a sheet tension test specimen measured using a small foil strain gauge, plotted versus the nominal stress. This plot exhibits a deviation from (apparently) linear behavior only after the local strain exceeded a value approximately 1.35 times the yield point strain determined by conventional tension tests (employing unnotched specimens).

From a design perspective the most serious problem associated with constrained yielding is that fracture may occur before local yielding is detectable. It is not unusual for fracture to occur, depending on the fracture
Time-dependent load-strain behavior observed here

Piobert-Lüder's bands observed here (but not below the open circle) (See Figure 9.17.)

Local yielding detectable here

$S_{nn, nom} = \frac{P}{A_{net}}$

Hysteresis loops observed here, relatively stable in size but ratcheting to the right as zero-to-max cyclic loading continued.

Note: The strain distribution changed as cyclic loading continued -- leading to the concept of stress "redistribution".

Local yielding starts here

Figure 9.16 Constrained yielding example. Plot of strain at the inside surface of a hole in a thin sheet versus the applied load. The strain-load plot deviated from linear only at strains well above the yield point strain. Piobert-Lüder's bands were observed only above the strain value indicated by the open circle. At lower strain values the strain field is inadequate to accommodate Piobert-Luder's band deformation.
Subsequent Piobert-Lüder's Bands

Initial Piobert-Lüder's band

\[ S_{nn} = \frac{P}{A} \]

Figure 9.17 Piobert-Lüder's band formation in (a) a tension test with a sheet specimen. The upper yield point (strength) corresponds to the initiation of the first Piobert-Lüder's band, a wedge of plastically deformed material in which the strain is abruptly equal to (approximately) the yield point elongation strain. Subsequently, other bands are initiated and fill the entire gauge portion of the specimen before the specimen exhibits a noticeable strain-hardening behavior. (b) A sheet specimen with a hole (See Figure 9.16).
toughness of the material and the notch configuration, at nominal stress values \( S_{nn,nom} = P/A_{net} \) equal to one-half to one-tenth the tensile yield strength. Thus local yielding, especially in thick members, dictates a design analysis in which both yield strength and fracture toughness (fracture mechanics) are given adequate consideration. (We illustrate only the yield strength analyses herein. Fracture mechanics is generally discussed in a mechanical metallurgy text.)