Homework 3:
2-8. Find the inverse z-transform using

i. power seriies
ii. inversion-formula
iii. partial fraction expansion
iv. discrete convolution

a. E(z) = $\dfrac{0.5z}{(z-1)(z-0.6)}$

(i) power series method:

$$\begin{array}{r|l} & 0.5z^{-1}+0.8z^{-2}+0.98z^{-3}+\dots \\ \hline z^2-1.6z+0.6 & 0.5z \\ & \underline{0.5z-0.8+0.3z^{-1}} \\ & 0.8-0.3z^{-1} \\ & \underline{0.8-1.28z^{-1}+0.48z^{-2}} \\ & 0.98z^{-1}-0.48z^{-2} \end{array}$$

e(0) = 0, e(1) = 0.5, e(2) = 0.8, …

(ii) inversion formula method

residue at a = $(z-a)E(z)z^{k-1}|_{z=a}$

$E(z)z^{k-1} = \dfrac{0.5z^k}{(z-1)(z-0.6)}$

$$e(k) = \left.\frac{0.5z^k}{(z-0.6)}\right|_{z=1} + \left.\frac{0.5z^k}{(z-1)}\right|_{z=0.6} = \frac{0.5}{(1-0.6)} + \frac{0.5(0.6)^k}{(0.6-1)} = 1.25(1-0.6^k)u(k)$$

(iii) partial fraction expansion:

$E(z)/z = \dfrac{0.5}{(z-1)(z-0.6)} = \dfrac{1.25}{z-1} + \dfrac{-1.25}{z-0.6}$

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$e(k) = 1.25(1 - 0.6^k)u(k)$

iv. discrete convolution

$$E_1(z) = \frac{0.5z}{z-0.6} \rightarrow e_1(k) = 0.5(0.6)^k$$

$$E_2(z) = \frac{1}{z-1} = z^{-1}\frac{z}{z-1} \rightarrow e_2(k) = u(k-1)$$

$e(0) = e_1(0)e_2(0)=0$
$e(1) = e_1(0)e_2(1)+ e_1(1)e_2(0) = 0.5$
$e(2) = e_1(0)e_2(2)+ e_1(2)e_2(0) + e_1(1)e_2(1) = 0.8$
…

b. $E(z) = \dfrac{0.5}{(z-1)(z-0.6)}$

(i) power series method:

$$
\begin{array}{r|l}
 & 0.5z^{-2} + 0.8z^{-3} + 0.98z^{-4} + \ldots \\
z^2 - 1.6z + 0.6 & 0.5 \\
 & \underline{0.5 - 0.8z^{-1} + 0.3z^{-2}} \\
 & 0.8z^{-1} - 0.3z^{-2} \\
 & \underline{0.8z^{-1} - 1.28z^{-2} + 0.48z^{-3}} \\
 & 0.98z^{-2} - 0.48z^{-3}
\end{array}
$$

$e(0) = 0$, $e(1) = 0$, $e(2) = 0.5$, …

(ii) inversion formula method

residue at $a = (z-a)E(z)z^{k-1}|_{z=a}$

$$E(z)z^{k-1} = \frac{0.5z^{k-1}}{(z-1)(z-0.6)}$$

$$e(k) = \left.\frac{0.5z^{k-1}}{(z-0.6)}\right|_{z=1} + \left.\frac{0.5z^{k-1}}{(z-1)}\right|_{z=0.6} =$$

$$= \frac{0.5}{(1-0.6)} + \frac{0.5(0.6)^{k-1}}{(0.6-1)} = 1.25 - 2.083(0.6^k), k \geq 1$$

(iii) partial fraction expansion:

$E_1(z) = \dfrac{0.5z}{(z-1)(z-0.6)} = \dfrac{1.25z}{z-1} + \dfrac{-1.25z}{z-0.6} \rightarrow e_1(k) = 1.25(1 - 0.6^k)$

$E(z) = z^{-1}E_1(z) \rightarrow e(k) = 1.25 - 1.25(0.6^{k-1})$

$e(k) = 1.25 - 2.083(0.6^k), k \geq 1$

$zE(z) = \dfrac{0.5z}{(z-1)(z-0.6)} = \dfrac{1.25z}{z-1} + \dfrac{-1.25z}{z-0.6}$

iv. discrete convolution

$$E_1(z) = \frac{0.5}{z-0.6} = z^{-1}\frac{0.5z}{z-0.6} \rightarrow e_1(k) = 0.5(0.6)^{k-1}u(k-1)$$

$$E_2(z) = \frac{1}{z-1} = z^{-1}\frac{z}{z-1} \rightarrow e_2(k) = u(k-1)$$

$e(0) = e_1(0)e_2(0)=0$
$e(1) = e_1(0)e_2(1)+ e_1(1)e_2(0) = 0.0$
$e(2) = e_1(0)e_2(2)+ e_1(2)e_2(0) + e_1(1)e_2(1) = 0.5$
…

c. $E(z) = \frac{0.5(z+1)}{(z-1)(z-0.6)}$

$e(0) = 0$
$e(k) = 2.5-3.33(0.6)^k$, $k\geq 1$
$E(z) = 0.5z^{-1} + 1.30z^{-2} + 1.78z^{-3} + 2.068z^{-4} + \dots$

d. $E(z) = \frac{z(z-0.7)}{(z-1)(z-0.6)}$

$e(k) = 0.75+0.25(0.6)^k$, $k\geq 1$
$E(z) = 1+ 0.9z^{-1} + 0.84z^{-2} + 0.804z^{-3} + \dots$

e. matlab verification:

```
num = [0 0 0.5]
dem = [1 –1.6 0.6]
[r,p,k] = residue(num,den)
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2-10.Solve difference equation,

$x(k) - 3x(k-1) + 2x(k-2) = e(k)$
if $k = 0,1 \rightarrow e(k) = 1$
if $k \geq 2 \rightarrow e(k) = 0$
$x(-2) = x(-1) = 0$

using

a. sequential technique

$x(0) = e(0) = 1$
$x(1) = e(1) + 3x(0) = 4$
$x(2) = e(2) + 3x(1) - 2x(0) = 10$
$x(3) = e(3) + 3x(2) - 2x(1) = 22$
…

b. z-tranform

$x(k) - 3x(k-1) + 2x(k-2) = e(k)$
$E(z) = 1 + z^{-1}$
$(1-3z^{-1} + 2z^{-2})X(z) = E(z) = 1 + z^{-1}$

$$X(z) = \frac{z^2}{(z-1)(z-2)}\frac{z+1}{z} = \frac{z(z+1)}{(z-1)(z-2)}$$

$$X(z) = \frac{Az}{z-1} + \frac{Bz}{z-2} = \frac{-2z}{z-1} + \frac{3z}{z-2}$$

$x(k) = -2 + 3(2^k)$

c. final value does not exist!

2-13. x(k+2) + 3x(k+1) =2x(k) = e(k)

if k = 0 → e(k) = 1

if k ≥ 1 → e(k) = 0

x(0) = 1

x(1) = -1

a. solve for x(k) as a function of k

x(k+2) + 3x(k+1) =2x(k) = e(k)

z^2 [X(z) – x(0) – x(1)z^{-1}] + 3z[X(z)-x(0)]+2X(z) = E(z) =1

(z^2 +3z + 2)X(z) = 1+ z^2 – z +3z

$$X(z) = \frac{z^2+2z+1}{z^2+3z+2} = \frac{z+1}{z+2}$$

$$X(z) = z\frac{z+1}{z(z+2)} = \frac{Az}{z} + \frac{Bz}{z+2} = 0.5 + \frac{0.5z}{z+2}$$

x(k) = 0.5 δ(k) + 0.5 $(-2)^k$

b. evaluate x(0), x(1) x(2) x(3)

x(0) = 1

x(1) = -1

x(2) = 2

x(3) = -4

c. verify (b) using power series method

$$\begin{array}{r|l} & 1 - z^{-1} + 2z^{-2}\ldots \\ z+2 & z+1 \\ & \underline{z+2} \\ & -1 \\ & \underline{-1-2z^{-1}} \\ & 2z^{-1} \\ & \underline{2z^{-1}} \end{array}$$

then x(0) = 1, x(1) = -1, x(2) = 2…

d. verify (b) by solving directly

x(k+2) + 3x(k+1) =2x(k) = e(k)

x(2) + 3x(1) =2x(0) =2

x(3) + 3x(2) =2x(1) = -4

2-18. (a-e), skip f,g
2nd order digital filters:

a. write the difference equation for fig 1, express y(k) as a function of y(k-i) & e(k-i)

$y(k) = -A_1y(k-1) - A_0y(k-2) + B_0e(k-2) + B_1e(k-1) + B_2e(k)$

b. dervie the filter TF Y(z)/E(z) by taking z-transform of (a)

$(1+A_1z^{-1} + A_0z^{-2})Y(z) = (B_2 + B_1z^{-1} + B_0z^{-2})E(z)$

$$\frac{B_2 + B_1 z^{-1} + B_0 z^{-2}}{1 + A_1 z^{-1} + A_0 z^{-2}}$$

c. write the difference equation for fig 2

$f(k) = e(k) - a_1f(k-1) - a_0f(k-2)$

$y(k) = -A_1y(k-1) - A_0y(k-2) + b_0f(k-2) + b_1f(k-1) + b_2f(k)$

d. dervie the filter TF Y(z)/E(z) by taking z-transform of (c) and eliminating F(z)

$F(z) = E(z) - (a_1z^{-1} + a_0z^{-2})F(z)$

$F(z) = E(z)/1+a_1z^{-1} + a_0z^{-2}$

$Y(z) = (b_0z^2 + b_1z + b_2) F(z)$

$Y(z) = ((b_0z^2 + b_1z + b_2)/(1+a_1z^{-1} + a_0z^{-2}))E(z)$

e. relate coefficients from (b) & (d) such that 2 filters realize same transfer function

$a_i = A_i, B_i = b_i$