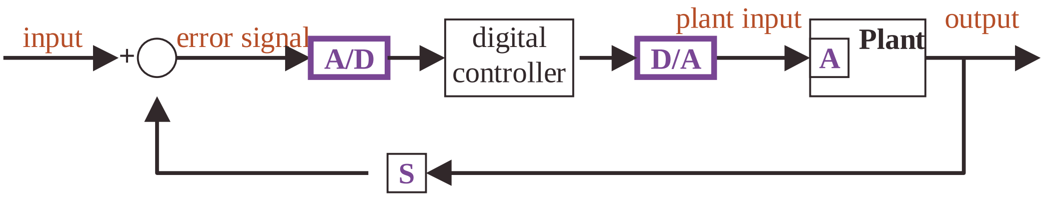
CH2: Disccrete Time Systems and Z Transforms

difference equations: used to describe Operation of Disccrete Time Systems
z-transform: transform used to analyze linear, time invarient, dicrete time systems

LTIDS: *linear, time invarient, dicrete time systems*
LTICS: *linear, time invarient, continuous time systems*

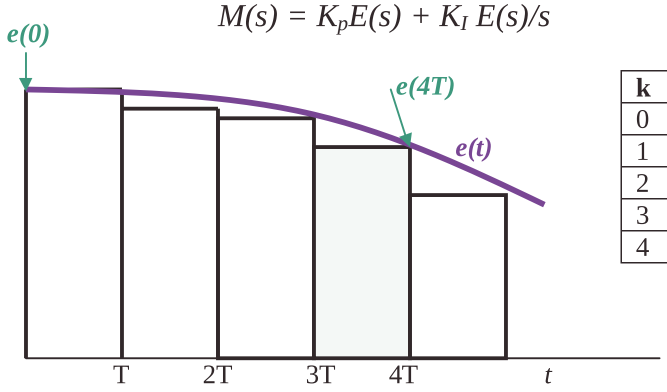
2.2: Discrete Time Systems: Controlled by digital computer
A/D conversion: used to convert continuous error signal
D/A conversion: used to convert digital signals to drive plant



Replace continuous proportional-intergral (PI) controller with digital system

$$m(t) = K_p e(t) + K_I \int_0^t e(\tau)d\tau$$

$$M(s) = K_p E(s) + K_I E(s)/s$$



k	x(kT)	e(kT)
0	Te(0)	e(0)
1	Te(0)+Te(T)	e(T)
2	T(e(0)+e(T)+e(2T))	e(2T)
3	T(e(0)+e(T)…+e(3T))	e(3T)
4	T(e(0)+e(T)…+e(4T))	e(4T)

Digital system must have same characteristics

- **rectangle rule for integration:**
  - *continuous:* $x(t) = \int_t e(\tau)d\tau$ **(2-1)**
  - *discrete:* $x(kT) = x((k-1)T) + Te(kT)$ **(2-2)**

- **digital controller** becomes 1st order linear difference equantion:

$$m(kT) = K_p e(kT) + K_I x(kT)$$

General form for 1st order linear difference equation

$$x(k) = b_1 e(k) + b_0 e(k-1) - a_0 x(k-1) \qquad \textbf{(2-3)}$$

$x(1) = b_1 e(1) + b_0 e(0) - a_0 x(0)$
$x(2) = b_1 e(2) + b_0 e(1) - a_0 x(1)$
$x(3) = b_1 e(3) + b_0 e(2) - a_0 x(2)$
…

General form for nth order linear difference equation

$$x(k) = \sum_{i=0}^{n} b_{n-i} e(k-i) - \sum_{i=1}^{n} a_{n-i} x(k-i) \qquad \textbf{(2-4)}$$

- can be used to model linear time invarient controller or entire system
- same equation for digitial filter
- implmented by digital computer

design issues

- select sampling interval, T
- select the order of the difference equation, n
- select filter coefficients for desired respones, a_i, b_i
- select enough bits to avoid significant round-off error

General form for nth order linear differential equation

$$x(k) = \sum_{i=0}^{n} \beta_{n-i} \frac{d^{n-i} e(t)}{dt^{n-i}} - \sum_{i=0}^{n-1} \alpha_{n-i} \frac{d^{n-i} y(t)}{dt^{n-i}} \qquad \textbf{(2-5)}$$

- can be used to model linear time invarient controller or entire system
- implemented with op amps & RC circuits (analog computer)
- same equation for analog filters

Approaches to designing digital compensators

1) design analog compensator then convert to digital compensators
2) direct design of digital compensators

2.3 Transform Methods

LTICS: use **Laplace Transform** for analysis and design

$$\frac{Y(s)}{E(s)} = \frac{\beta_n s^n + \beta_{n-1} s^{n-1} + \ldots + \beta_1 s + \beta_0}{\alpha_n s^n + \alpha_{n-1} s^{n-1} + \ldots + \alpha_1 s + \alpha_0} \qquad \textbf{(2-6)}$$

LTIDS: transform for difference equations

ordinary z-transform: **define** $E(z)$ as power series in terms z^{-k} with coefficents = $\{e(k)\}$

‘{}’ denotes application to sequence

$e(k)$ implies $e(kT)$: drop T for convenience

$$E(z) = Z[\{e(k)\}] = \sum_{k=0}^{\infty} e(k) z^{-k} = e(0) + e(1)z^{-1} + e(2)z^{-2} + \ldots. \qquad \textbf{(2-7)}$$

$$e(k) = Z^{-1}[E(Z)] = \frac{1}{2\pi j} \oint_{\Gamma} E(z) z^{k-1} dz, \; j = \sqrt{-1}$$

Z indicates z transform

$\mathbf{Z}^{-1}$ indicates inverse z-transform

$$E(z) = \sum_{k=0}^{\infty} e(k) z^{-k} \qquad \textbf{(2-8)}$$

double-sided z-transform (Generating Function)

$$G[\{f(k)\}] = \sum_{k=-\infty}^{\infty} f(k) z^{-k} \qquad \textbf{(2-9)}$$

Examples

1. $E(z) = \mathbf{1} + \mathbf{3}z^{-1} - \mathbf{2}z^{-2} + z^{-4} \ldots$

   $\mathbf{e(0)} = 1, \quad \mathbf{e(1)} = 3, \quad \mathbf{e(2)} = -2, \quad \mathbf{e(3)} = 0, \quad \mathbf{e(4)} = 1 \ldots$

2. if $e(k) = 1$ for all k → $E(z) = 1 + z^{-1} + z^{-2} + \ldots$

   $$\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i \qquad |x| < 1 \rightarrow \quad E(z) = \frac{1}{1-z^{-1}} = \frac{z}{z-1}, \qquad |z^{-1}| < 1$$

   - transform of any time function with value = 1 every T seconds
   - unit step

3. $e(k) = \varepsilon^{-akT}$ →

   $$E(z) = 1 + \varepsilon^{-aT} z^{-1} + \varepsilon^{-a2T} z^{-2} + \varepsilon^{-a3T} z^{-3} + \ldots$$
   $$= 1 + (\varepsilon^{-aT} z^{-1}) + (\varepsilon^{-aT} z^{-1})^2 + (\varepsilon^{-aT} z^{-1})^3 + \ldots$$

   $$x = \varepsilon^{-aT} z^{-1} \rightarrow \quad E(z) = \frac{1}{1-\varepsilon^{-aT} z^{-1}} = \frac{z}{z-\varepsilon^{-aT}}, \qquad |\varepsilon^{-aT} z^{-1}| < 1$$

   $e(k)$ generated by sampling the function $e(t) = \varepsilon^{-akT}$ at intervals of k

2.4 Properties of z-transforms

Unit step function

$if\ k < 0 \rightarrow u(k) = 0$

$if\ k \geq 0 \rightarrow u(k) = 1$

$$\forall k < 0 \rightarrow e(k) = 0 : Z[e(k)] = Z[e(k)u(k)]$$

$$e(k-n)u(k-n) = e(k)u(k)\,|_{k=k-n}$$
$$e(k+n)u(k+n) = e(k+n)u(k)$$

DiscreteUnit Impulse

if $k=n \rightarrow \delta(k\text{-}n) = 1$

if $k \neq n \rightarrow \delta(k\text{-}n) = 0$

$$Z[\delta(k-n)] = \sum_{k=0}^{\infty} \delta(k-n) z^{-k} = z^{-n}$$

(1) Addition & Subtraction: $Z[e_1(k) \pm e_2(k)] = E_1(z) \pm E_2(z)$

$$Z[e_1(k) \pm e_2(k)] = \sum_{k=0}^{\infty} (e_1(k) \pm e_2(k)) z^{-k}$$
$$= \sum_{k=0}^{\infty} e_1(k) z^{-k} \pm \sum_{k=0}^{\infty} e_2(k) z^{-k}$$
$$= E_1(z) \pm E_2(z)$$

(2) Multiplication by a Constant: $Z[ae(k)] = aZ[e(k)] = aE(z))$

$$aZ[e(k)] = \sum_{k=0}^{\infty} ae(k) z^{-k} = a\sum_{k=0}^{\infty} e(k) z^{-k} = aE(z)$$

→ *Linearity follows from (1) & (2)*

(3)Real Translation:

(i) time delayed: $Z[e(k-n)u(k-n)] = z^{-n}E(z)$

$$
\begin{aligned}
&Z[e(k-n)u(k-n)] = e(0)z^{-n} + e(1)z^{-(n+1)} + e(2)z^{-(n+2)} + \ldots \\
&= z^{-n}(e(0) + e(1)z^{-1} + e(2)z^{-2} + \ldots) \\
&= z^{-n}E(z)
\end{aligned}
$$

(ii) time forwarded: $Z[e(k+n)u(k)] = z^{n}[E(z) - \sum_{k=0}^{n-1} e(k)z^{-k}]$

$$
\begin{aligned}
&Z[e(k+n)u(k)] = e(n) + e(n+1)z^{-1} + e(n+2)z^{-2} + \ldots \\
&= z^{n}[e(0) + e(1)z^{-1} + e(2)z^{-2} + \ldots \\
&+ e(n-1)z^{-(n-1)} + e(n)z^{-n} + +e(n+1)z^{-(n+1)} + e(n+2)z^{-(n+2)} + \ldots \\
&+ \ldots - e(0) - e(1)z^{-1} - e(2)z^{-2} - \ldots - e(n-1)z^{-(n-1)}] \\
&= z^{n}[E(z) + \sum_{k=0}^{n-1} e(k)z^{-k}]
\end{aligned}
$$

shifting examples:

k	e(k)	e(k-1)	e(k-2)	e(k+2)
0	1	0	0	1.8
1	1.5	1	0	1.9
2	1.8	1.5	1	2.0
3	1.9	1.8	1.5	2.5
4	2.0	1.9	1.8	-
4	2.5	2.0	1.9	-

(4) Complex Translation: $Z[\varepsilon^{ak}e(k)] = E(z\varepsilon^{-a})$

$$
\begin{aligned}
Z[\varepsilon^{ak}e(k)] &= e(0)+\varepsilon^{a}e(1)z^{-1}+\varepsilon^{a2}e(2)z^{-2}+\varepsilon^{a3}e(3)z^{-3}+\ldots \\
&= e(0)+e(1)(\varepsilon^{-a}z)^{-1}+e(2))(\varepsilon^{-a}z)^{-2}+e(3))(\varepsilon^{-a}z)^{-3}+\ldots \\
&= E(z)\big|_{z=\varepsilon^{-a}z} = E(z\varepsilon^{-a})
\end{aligned}
$$

(5) Initial Value: $e(0) = \underset{\lim z\rightarrow\infty}{E(z)}$

$e(0) + e(1)\infty^{-1} + \ldots = e(0)$

(6) Final Value: $\lim\limits_{n\to\infty} e(n) = \lim\limits_{z\to 1}(z-1)E(z)$

$$Z[e(k+1)-e(k)] = \lim_{n\to\infty}\left[\sum_{k=0}^{\infty} e(k+1)z^{-k} - \sum_{k=0}^{\infty} e(k)z^{-k}\right]$$

$$= \lim_{n\to\infty}\left[-e(0)+e(1)(1-z^{-1})+e(2)(z^{-1}-z^{-2})+\ldots+e(n)(z^{-n+1}-z^{-n})+e(n+1)z^{-n}\right]$$

$$\to \lim_{z\to 1} Z[e(k+1)-e(k)] = \lim_{n\to\infty}[e(n+1)-e(0)]$$

$$\to Z[e(k+1)-e(k)] = Z[e(k+1)] - Ze(k) = zE(z)-e(0)-E(z) = (z-1)E(z)-ze(0)$$

$$\therefore \lim_{z\to 1}(z-1)E(z)-ze(0) = \lim_{n\to\infty}[e(n+1)-e(0)] \to \lim_{n\to\infty} e(n) = \lim_{z\to 1}(z-1)E(z)$$

* *provided left - side limit exists (all poles of E(z) are in unit circle*
except for possible simple pole at z = 1)

$$ie1.\ Z[\varepsilon^{-akT}] = \frac{z}{z-\varepsilon^{-aT}}$$

$$\rightarrow Z[\varepsilon^{-a(k-5)T}u(k-5)] = z^{-5}\frac{z}{z-\varepsilon^{-aT}}$$

$$\rightarrow Z[\varepsilon^{-a(k+3)T}u(k)] = z^{3}\left[\frac{z}{z-\varepsilon^{-aT}} - 1 - \varepsilon^{-aT}z^{-1} - \varepsilon^{-a2T}z^{-2}\right]$$

ie 2: *by definition:* $Z\{e(k)\} = \sum_{k=0}^{\infty} e(k)z^{-k}$

then $Z\{ke(k)\} = \sum_{k=0}^{\infty} ke(k)z^{-k}$

$$= -z\sum_{k=0}^{\infty} -kz^{-k-1}e(k)$$

$$= -z\frac{dE(z)}{dz} = -z\left(\frac{1}{z-1} - \frac{z}{(z-1)^2}\right) = -z\left(\frac{-1}{(z-1)^2}\right)$$

$$= \frac{z}{(z-1)^2}$$

and $Z\{k\varepsilon^{ak}\} = \left.\frac{z}{(z-1)^2}\right|_{z=z\varepsilon^{-a}} = \frac{z\varepsilon^{-a}}{(z\varepsilon^{-a}-1)^2} = \frac{\varepsilon^{a}z}{(z-\varepsilon^{a})^2}$

$$ie3.\ if\ \ e(k) = 1 \rightarrow E(z) = \frac{z}{z-1}$$

$$e(0) = \lim_{z\to\infty}\frac{z}{z-1} = \lim_{z\to\infty}\frac{1}{(1-z^{-1})} = 1$$

$$\lim_{k\to\infty} e(k) = \lim_{z\to 1}(z-1)E(z) = \lim_{z\to 1} z = 1$$

Properties of Z-transforms

sequence	*transform*	
$e(k)$	$E(z)=\sum_{k=0}^{\infty} e(k)z^{-k}$	
$a_1e_1(k)+a_2e_2(k)$	$a_1E_1(z)+a_2E_2(z)$	
$e(k-n)u(k-n)\vert_{n\geq 0}$	$z^{-n}E(z)$	
$e(k+n)u(k)$	$z^n\left[E(z)-\sum_{k=0}^{n-1} e(k)z^{-k}\right]$	
$\varepsilon^{ak}e(k)$	$E(z\varepsilon^{-a})$	
$ke(k)$	$-z\frac{dE(z)}{dz}$	
$e_1(k)*e_2(k)$	$E_1(z)E_2(z)$	
$e_1(k)=\sum_{n=0}^{k} e(n)$	$E_1(z)=\frac{z}{z-1}E(z)$	
$e(0)$	$\lim_{z\to\infty} E(z)$	
$\lim_{k\to\infty} e(k)$	$\lim_{z\to 1}(z-1)E(z)$	(if $e(\infty)$ exists)

z-Transform Table

Sequence	z - Transform
$\delta(k - n)$	z^{-n}
a^k	$\frac{z}{z - a}$
k	$\frac{z}{(z - 1)^2}$
k^2	$\frac{z(z+1)}{(z - 1)^3}$
ka^k	$\frac{az}{(z - a)^2}$
$\sin ak$	$\frac{z \sin a}{z^2 - 2z\cos a + 1}$
$\cos ak$	$\frac{z(z - \cos a)}{z^2 - 2z\cos a + 1}$
$a^k \sin bk$	$\frac{az \sin b}{z^2 - 2az\cos b + a^2}$
$a^k \cos bk$	$\frac{z^2 - az\cos b}{z^2 - 2az\cos b + a^2}$

2.5 Solutions to Difference Equations

3 basic techniques

1. *classical approach*: find complimentary & particular parts of soln (~ *differential equations*)
2. *sequential procedure*: used in digital computer solutions
3. *z -transform*

<u>1. classical approach</u> not discussed

<u>2. sequential procedure</u>

ie: find $m(k)$ for $m(k) = e(k) - e(k-1) - m(k-1)$ $\qquad k \geq 0,$

if $k = 1,3,5\ldots$ → $e(k) = 1$
if $k = 0,2,4,6\ldots$ → $e(k\ = 0$
if $k < 0$ → $e(k) = m(k)\ = 0$

k	*m(k)*	*e(k)*	*e(k-1)*	*m(k-1)*
0	0	0	0	0
1	1	1	0	0
2	-2	0	1	1
3	3	1	0	-2
4	-4	0	1	3
5	5	1	0	-4

<u>3. z-transform</u>

$$m(k) + a_{n-1}m(k-1) + a_{n-2}m(k-2) + \ldots + a_0m(k-n) = b_ne(k) + b_{n-1}e(k-1) + \ldots + b_0e(k-n) \qquad \textbf{(2-19)}$$

$$(1 + a_{n-1}z^{-1} + a_{n-2}z^{-2} + \ldots + a_0z^{-n})\,M(z) = (b_n + b_{n-1}z^{-1} + \ldots + b_0z^{-n})E(z) \qquad \textbf{(2-20)}$$

$$\rightarrow \quad M(z) = \frac{(b_n + b_{n-1}z^{-1} + \ldots + b_0z^{-n})}{(1 + a_{n-1}z^{-1} + a_{n-2}z^{-2} + \ldots + a_0z^{-n})}\,E(z) \qquad \textbf{(2-21)}$$

$$\rightarrow m(k) = Z^{-1}[M(z)]$$

with **non-zero initial conditions:** replace k with $k+n$ → *include* $m(0), m(1),\ldots,m(n-1)$

$$m(k+n) + a_{n-1}m(k-1+n) + \ldots + a_0m(k) = b_ne(k+n) + b_{n-1}e(k-1+n) + \ldots + b_0e(k) \qquad \textbf{(2-22)}$$

$$Z[m(k+i)] = z^i[M(z) - m(0) - m(1)z^{-1} - m(2)z^{-2} - \ldots - m(i-1)z^{-(i-1)} \qquad \textbf{(2-23)}$$

ie

$$m(k) = e(k) - e(k-1) - m(k-1),$$
$$k = 1,3,5\ldots \rightarrow e(k) = 1$$
$$k = 0,2,4,6\ldots \rightarrow e(k) = 0$$
$$(1 + z^{-1})M(z) = (1 - z^{-1})E(z)$$
$$M(z) = \frac{1 - z^{-1}}{1 + z^{-1}}E(z) = \frac{z-1}{z+1}E(z)$$

$$E(z) = 1 + z^{-2} + z^{-4} + z^{-6} + \ldots = 1 + (z^2)^{-1} + (z^2)^{-2} + (z^2)^{-3}\ldots$$
$$= \frac{1}{1 - z^{-2}} = \frac{z^2}{z^2 - 1} = \frac{z^2}{(z-1)(z+1)}$$

$$M(z) = \frac{z-1}{z+1}\frac{z^2}{(z-1)(z+1)} = \frac{z^2}{z^2 + 2z + 1}$$
$$\text{expand into power series} \rightarrow z^2 + 2z + 1 \overline{\big)\, z^2} = 1 - 2z^{-1} + 3z^{-2} - 4z^{-3}\ldots$$

$$M(z) = 1 - 2z^{-1} + 3z^{-2} - 4z^{-3}\ldots = \sum_{k=0}^{\infty} m(k)z^{-k}$$

2.6 Inverse Z-Transform: recover e(k) from E(z)

1. Power Series Method: convert z-transform to power series: $e_0 + e_1z^{-1} + e_2z^{-2} + e_3z^{-3}\ldots$

$$E(z) = \frac{E_N(z)}{E_D(z)} \rightarrow E_D(z)\overline{\big)\,E_N(z)}$$

may be possible to recognize *e(k)* from *e(0)* , *e(1), e(2), …*

$z^{-1} + 10z^{-3} + 100z^{-5} + 1000z^{-7}\ldots$

$z^2 - 10 \,\big|\, z$

$\underline{z - 10z^{-1}}$

$10z^{-1}$

$\underline{10z^{-1} - 100z^{-3}}$

$100z^{-3}$

$\underline{100z^{-3} - 1000z^{-5}}$

$1000z^{-5}$

e(1) = 1, e(3) = 10, e(5) = 100, e(7) = 1000
if *k* = odd → *e(k) = 10e(k-2)*
if k = event → *e(k) = 0*

2. Partial Fraction Expansion (~similar to Inverse Laplace Transform)

- expand into partial fractions
- use table of *z-transform pairs (appendix VIII)* to determine inverse

- ***exponential*** **commonly encountered z-transform:** a^k

  $Z(a^k) = 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + \ldots \rightarrow \dfrac{1}{1-az^{-1}} = \dfrac{z}{z-a}$

- ***z*** appears in the numerator of transform → perform partial fraction expansion on $\dfrac{E(z)}{z}$

(1) **real poles**

ie:

$$E(z)=\frac{z}{(z-5)(z-2)} \rightarrow \frac{E(z)}{z}=\frac{1}{(z-5)(z-2)}=\frac{-1}{z-5}+\frac{-3}{z-2}$$

$$Z^{-1}[E(z)]=Z^{-1}\left[\frac{-z}{z-1}\right]+Z^{-1}\left[\frac{-3z}{z-2}\right]$$

$$e(k)=-1-3(2^k)$$

ie:

$$E_1(z)=z^{-1}E(z) \rightarrow \frac{E_1(z)}{z}=\frac{1}{z(z-5)(z-2)}=\frac{a}{z}+\frac{b}{z-5}+\frac{c}{z-2}$$

$$a=\frac{1}{(z-5)(z-2)}\Big|_{z=0}=0.1, \quad b=\frac{1}{z(z-2)}\Big|_{z=5}=0.067, \quad c=\frac{1}{z(z-5)}\Big|_{z=2}=-0.167$$

$$E_1(z)=0.1+\frac{0.067z}{z-5}-\frac{0.167z}{z-2}$$

$$Z^{-1}[0.1]=0.1\delta(k) \qquad \{\delta(k)=1 \text{ if } k=0, \text{ otherwise } 0\}$$

$$Z^{-1}\left[\frac{0.067z}{z-5}\right]=0.067(5^k)$$

$$Z^{-1}\left[\frac{0.167z}{z-2}\right]=0.167(2^k)$$

ie:

$$E_2(z)=z^{-2}E(z) \rightarrow e_2(k)=Z^{-1}\{z^{-2}E(z)\}=e(k-2)u(k-2)$$

$$e_2(k)=-1-3(2^{k-2})u(k-2)$$

$$\text{if } k \geq 2 \rightarrow e_2(k)=-1-3(2^{k-2})$$

$$\text{if } k<2 \rightarrow e_2(k)=0$$

2) complex poles

$$y(k)=A\varepsilon^{akT}\cos(bkT+\theta)=\frac{A\varepsilon^{akT}}{2}\left[\varepsilon^{jbkT}\varepsilon^{j\theta}+\varepsilon^{-jbkT}\varepsilon^{-j\theta}\right]$$

$$=\frac{A}{2}\left[\varepsilon^{(aT+jbT)k}\varepsilon^{j\theta}+\varepsilon^{(aT-jbT)k}\varepsilon^{-j\theta}\right]$$

$$Y(z)=\frac{A}{2}\left[\frac{\varepsilon^{j\theta}z}{z-\varepsilon^{(aT+jbT)k}}+\frac{\varepsilon^{-j\theta}z}{z-\varepsilon^{(aT-jbT)k}}\right] \qquad \textit{(from appendix 8)}$$

$$=\frac{0.5A\varepsilon^{j\theta}z}{z-\varepsilon^{(aT+jbT)k}}+\frac{0.5A\varepsilon^{-j\theta}z}{z-\varepsilon^{(aT-jbT)k}}=\frac{k_1z}{z-p_1}+\frac{k_1^*z}{z-p_1^*}$$

- where $k_1 = 0.5A\varepsilon^{j\theta}$ and $p_1 = \varepsilon^{(aT+jbT)k}$
- solve for discrete time function
  - determine aT and bT from poles
  - determine A and θ from partial fraction expansion

$$p_1=\varepsilon^{aT}\varepsilon^{jbT}=\varepsilon^{aT}\angle bT \rightarrow aT=\ln|p_1|,\ \ bT=\arg\ p_1$$

$$k_1=\frac{A\varepsilon^{j\theta}}{2}=\frac{A}{2}\angle\theta \rightarrow A=2|k_1|,\ \ \theta=\arg k_1$$

$$a+jb \rightarrow magnitude=\sqrt{a^2+b^2},\quad \angle=\tan^{-1}\frac{b}{a}$$

ie: determining coefficients in partial fraction expansions

$$Y(z)=\frac{-3.894z}{z^2+0.6065}=\frac{-3.894z}{(z+j0.7788)(z-j0.7788)}=\frac{k_1z}{z-j0.7788}+\frac{k_1^*z}{z+j0.7788}$$

$$\frac{Y(z)}{z}=\frac{k_1}{z-j0.7788}+\frac{k_1^*}{z+j0.7788}$$

$$\rightarrow k_1=\frac{-3.894}{(z+j0.7788)}\Big|_{z=j0.7788}=\frac{-3.894}{2(j0.7788)}=\left|\frac{3.894}{1.556}\right|\tan^{-1}\tfrac{b}{0}=2.5\angle 90$$

$$\rightarrow k_1^*=\frac{-3.894}{(z-j0.7788)}\Big|_{z=-j0.7788}=\frac{3.894}{2(j0.7788)}=\left|\frac{3.894}{1.556}\right|\tan^{-1}\tfrac{b}{0}=2.5\angle 90$$

$$\rightarrow p_1=j0.7788$$

$$aT=\ln|p_1|=-0.25$$

$$bT=\arg\ p_1=\pi/2$$

$$A=2|k_1|=5$$

$$\theta=\arg k_1=\pi/2$$

$$y(k)=A\varepsilon^{akT}\cos(bkT+\theta)=5\varepsilon^{-0.25k}\cos\left(\tfrac{\pi}{2}k+\tfrac{\pi}{2}\right)=5\varepsilon^{-0.25k}\sin(\tfrac{\pi}{2}k)$$

- **residues**: *partial fraction expansion coefficients* k_1, k^*_1 for poles of Y(z)
- *k* has vaules if numerator of Y(z) w/ higher order than denominator

3. Inversion Formula Method: most general technique

$$e(k) = \frac{1}{2\pi j} \oint_{\Gamma} E(z) z^{k-1} dz \qquad \textbf{(2-31)}$$

line integral in z-*plane*
along any closed path, Γ, that encircles all finite poles of $E(z)z^{k-1}$

$$e(k) = \sum_{\text{@poles of } E(z)z^{k-1}} [\text{residues of } E(z)z^{k-1}] \qquad \textbf{(2-32)}$$

simple pole at $z = a$: $\quad residue_{z=a} = (z-a)E(z)z^{k-1}|_{z=a}$ **(2-33)**

pole of order *m* at $z = a$: $\quad residue_{z=a} = \dfrac{1}{(m-1)!} \dfrac{d^{m-1}}{dz^{m-1}} (z-a)^m E(z)z^{k-1}|_{z=a}$ **(2-34)**

ie:

$$E(z) = \frac{z}{(z-1)(z-2)} \rightarrow e(k) = \frac{z^k}{(z-2)}|_{z=1} + \frac{z^k}{(z-1)}|_{z=2} = -1 + 2^k$$

$$E_1(z) = \frac{1}{(z-1)(z-2)} \quad \rightarrow \frac{E_1(z)}{z} = \frac{1}{z(z-1)(z-2)}$$

approach 1: from real translation property: $e_1(k) = e(k-1)u(k-1) \;\; [-1 + 2^{k-1}]\, u(k-1)$

approach 2: from inversion property: $E_1(z)z^{k-1} = \dfrac{z^{k-1}}{(z-1)(z-2)}$

pole at $z = 0$ only for $k = 0$: $e(0) = \displaystyle\sum_{z=0,1,2} residues \frac{1}{z(z-1)(z-2)} = 0.5 - 1 + 0.5 = 0$

values for $e(k)$, $k \geq 0$ obtained from 2-32

ie: multi-order poles

$$E(z) = \frac{z}{(z-1)^2} \rightarrow e(k) = \frac{1}{(2-1)!} \frac{d}{dz} \left[(z-1)^2 \frac{z}{(z-1)^2} z^{k-1} \right]_{z=1}$$

$$= \frac{dz^k}{dz} = kz^{k-1}|_{z=1} = k$$

4. Discrete Convolution (analogous to continuous convolution)

suppose $E(z)$ can be expressed as: $E(z) = E_1(z)E_2(z)$ **(2-35)**

let $E_1(z)$ and $E_2(z)$ be expressed a power series, then

$E(z) = [e_1(0) + e_1(1)\mathbf{z}^{-1} + e_1(2)\mathbf{z}^{-2} + e_1(3)\mathbf{z}^{-3}]\,[e_2(0) + e_2(1)\mathbf{z}^{-1} + e_2(2)\mathbf{z}^{-2} + e_2(3)\mathbf{z}^{-3}]$

$E(z) = e_1(0)e_2(0) + [e_1(0)e_2(1) + e_1(1)e_2(0)]\mathbf{z}^{-1} + [e_1(0)e_2(2) + e_1(1)e_2(1) + e_1(2)e_2(0)]\mathbf{z}^{-2}\ldots$

$e(k) = e_1(0)e_2(k) + e_1(1)e_2(k-1) + \ldots + e_1(k)e_2(0)$

$e(k) = \sum_{n=0}^{k} e_1(n)e_2(k-n) = \sum_{n=0}^{k} e_1(k-n)e_2(n)$

$e(k) = Z^{-1}[E_1(z)E_2(z)] = e_1(k)*e_2(k)$ **(2-39)**

ie:

$$E(z) = \frac{z}{(z-2)(z-5)} = E_1(z)E_2(z) = \frac{z}{(z-2)}\frac{1}{(z-5)}$$

$$= \sum_{k=0}^{\infty}(2z^{-1})^k \cdot z^{-1}\sum_{k=0}^{\infty}(5z^{-1})^k$$

$$= \sum_{k=0}^{\infty}(2^k z^{-k}) \cdot \sum_{k=0}^{\infty}(5^k z^{-k-1})$$

$$= (1 + 2z^{-1} + 2^2 z^{-2} + 2^3 z^{-3} + 2^4 z^{-4}\ldots)(z^{-1} + 5z^{-2} + 5^2 z^{-3} + 5^3 z^{-4}\ldots)$$

$$= z^{-1} + (5+2)z^{-2} + (5^2 + 10 + 2^2)z^{-3} + (5^3 + 2\cdot 5^2 + 2^2\cdot 5 + 2^3)z^{-4}$$

$$e(k) = \sum_{n=0}^{k} e_1(n)e_2(k-n)$$

$$e(4) = \sum_{n=0}^{4} e_1(n)e_2(4-n) = e_1(0)e_2(4) + e_1(1)e_2(3) + e_1(2)e_2(2) + e_1(3)e_2(1) + e_1(4)e_2(0)$$

$$= 1\cdot 5^3 + 2\cdot 5^2 + 2^2\cdot 5 + 2^3\cdot 1 + 2^4\cdot 0 = 203$$

2.7 Simulation Diagrams & Flow Graphs

Linear Time Invarient System can be represented by

- difference equation
- transfer function
- simulation diagram
- flow graphs

(1) Simulation Diagram:

Simulation Digram Elements

- to obtain *TF* of block diagram → use *block diagram reduction* or *Mason's Gain Formula*

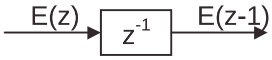
i. Shift Register = Delay element


shift register


ideal delay element

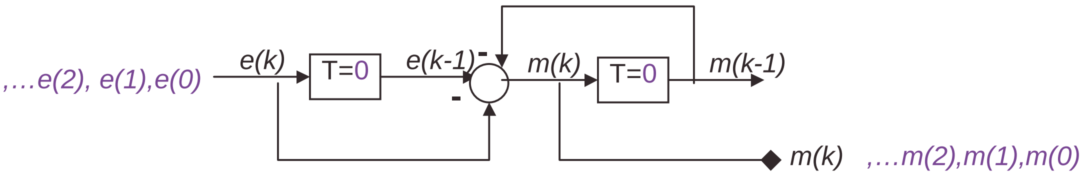
- simulation of *continuous systems* → basic element is integrator
- simulation of *discrete systems* → basic element is delay element *(memory)*
  - transform of input: $Z[e(k)]=E(z)$
  - transform of output: $Z[e(k-1)] = z^{-1}E(z)$

transfer function of time delay element = z^{-1}


ideal delay element

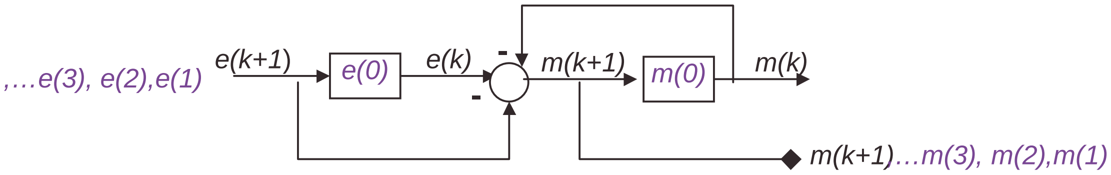
ie: system representd by difference equation: $m(k) = e(k) - e(k-1) - m(k-1)$

- input = $e(k)$
- ouput = $m(k)$
- use ASICs, FPGAs to implement specific equation using *shift registers, adders, summers…*
  - set all registers to 0 *(initial conditions = 0)*
    - $m(0) = 0$
    - $e(0) = 0$
  - at time $t = 0, 2T, 4T, \ldots$ → $e(kT) = 1$
  - at time $t = 1, 3T, 5T, \ldots$ → $e(kT) = 0$
  - output $m(k)$ available at time $t=kT$



non-zero initial conditions:

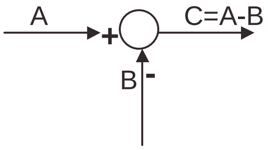
- replace k with $k+1$: $m(k+1) = e(k+1) - e(k) - m(k)$
- using register for $e(k)$ → initialize $e(0)$
- using register for $m(k)$ → initialize $m(0)$

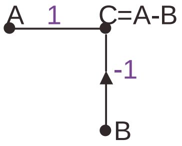


Transfer Function: $(1+z^{-1})M(z) = (1-z^{-1})E(z)$ → $\dfrac{M(z)}{E(z)} = \dfrac{1-z^{-1}}{1+z^{-1}}$

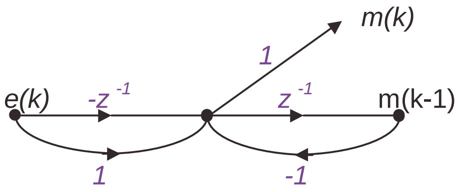
(2) Signal Flow Graph

- basic elements are *branches* & *nodes*
- output signal = branch gain * input signal
- signal at a node = sum of all signals from branches coming into node
- to obtain *TF* of signal flow graph → use Mason's Gain Formula





ie: $m(k) = e(k) - e(k-1) - m(k-1)$



Transfer Function *(Mason's Gain Formula)*

$M_1 = -z^{-1}$

$M_2 = 1$

$L_1 = -z^{-1}$

$\Delta = 1 - (L1) = 1 + z^{-1}$

$\Delta_1 = 1$

$\Delta_2 = 1$

$$T = \frac{(1 - z^{-1})}{(1 + z^{-1})}$$

Signal Flow Graph		**Block Diagram**
branch	↔	block
node	↔	signal
input node	↔	input signal
output node	↔	output signal

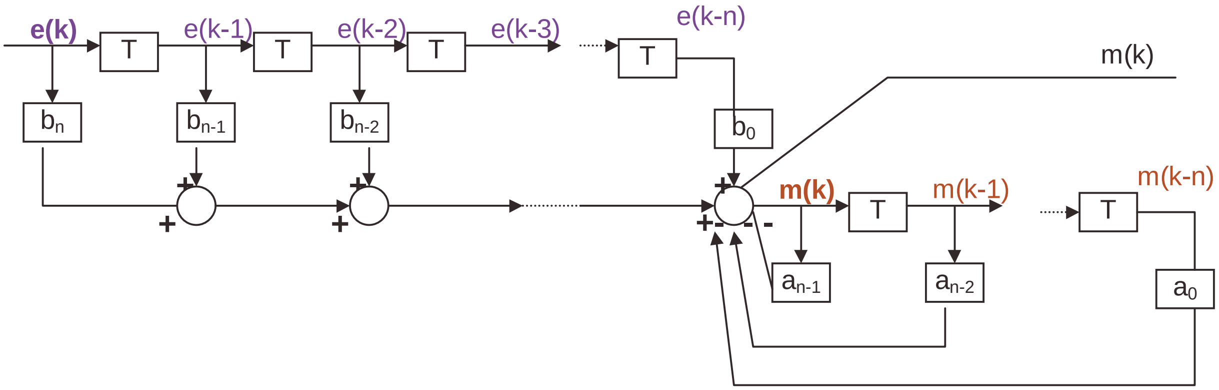
General nth order diffence equation

$$m(k) + a_{n-1}m(k-1) +\ldots+ a_0 m(k-n) = b_n e(k) + b_{n-1}e(k-1)+\ldots+ b_0 e(k-n) \tag{2-41}$$

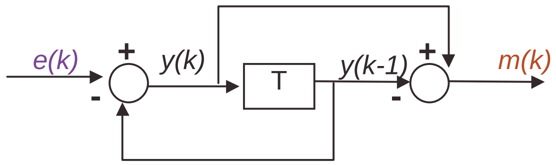
$$M(z) + a_{n-1} z^{-1} M(z) +\ldots+ a_0\, z^{-n} M(z) = b_n E(z) + b_{n-1} z^{-1}E(z)+\ldots+ b_0 z^{-n}E(z) \tag{2-42}$$

$$\frac{M(z)}{E(z)} = \frac{b_n + b_{n-1} z^{-1} +\ldots+ b_0 z^{-n}}{1 + a_{n-1} z^{-1} +\ldots+ a_0\, z^{-n}} \tag{2-43}$$

Non-minimal representation: minimal representaion needs only *n* delay elements



Example of minimal simulation diagram

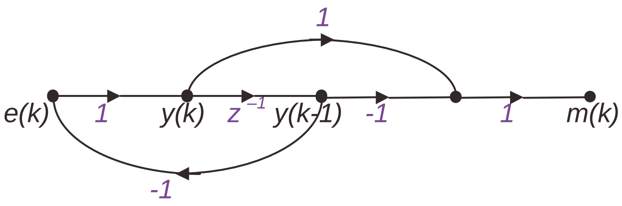


determine transfer function

1. $m(k) = y(k) – y(k\text{-}1)$ → $M(z) = (1\text{-}z^{-1})Y(z)$
   $y(k) = e(k)\text{-}y(k\text{-}1)$ → $E(z) = (1+z^{-1})Y(z)$

$$\frac{M(z)}{E(z)} = \frac{1-z^{-1}}{1+z^{-1}}$$

2. Masons Gain Formula



$M_1 = - z^{-1}$
$M_2 = 1$
$L_1 = -z^{-1}$
$\Delta = 1 + z^{-1}$
$\Delta_1 = 1$
$\Delta_2 = 1$

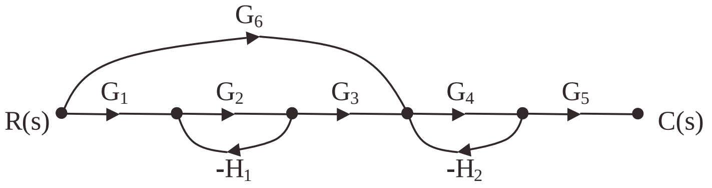
$$T = \frac{M_1\Delta_1 + M_2\Delta_2}{\Delta} = \frac{1-z^{-1}}{1+z^{-1}}$$

Review of Masons Gain Formula: gives transfer function from source to sink node

Source Node: signals only flow away from
Sink Node: signals only flow towards
Path: sequence of uni-directinal branches
Loop: closed path w/ no node encountered more than once, each branch w/ *in* & *out* branch
Forward Path: connects source to sink
Path Gain: product of transfer functions of all branches in the path
Loop Gain: product of transfer functions of all branches in the loop
Nontouching Loops: two loops with no nodes in common
Nontouching Loop & Path: loop and path with no nodes in common

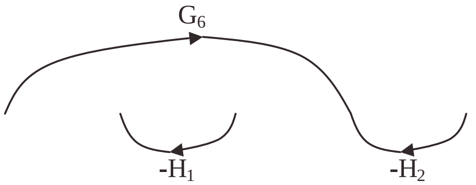
$$T = \frac{1}{\Delta}\sum_{k=1}^{p} M_k \Delta_k = \frac{1}{\Delta}(M_1\Delta_2 + \ldots + M_p\Delta_p)$$

T: transfer function
L_i: loop gain of ith loop
Δ: 1-sum(all individual loop gains)
 + sum(product of loop gains of all combinations non-touching loops, 2 at a time)
 - sum(product of loop gains of all combinations non-touching loops, 3 at a time)
 + sum(product of loop gains of all combinations non-touching loops, 4 at a time)
 …
M_k: path sign of kth forward path
Δ_k: value of Δ for part of flow graph not touching kth forward path



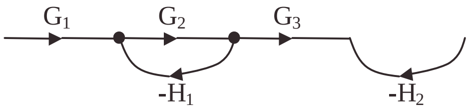
(i) two non-touching loops:
$L_1 = -G_2H_1$
$L_2 = -G_4H_2$

(ii) two froward paths
$M_1 = G_1G_2G_3G_4G_5$ – touches both loops
$M_2 = G_6G_4G_5$ – touches only $-G_4H_2$



$\Delta = 1 - (L_1 + L_2) + (L_1L_2)$
$\Delta_i = \Delta$ of flow graph without M_i (and any nodes)
$\Delta_1 = 1$
$\Delta_2 = 1 - L_1$

$$T = \frac{G_1G_2G_3G_4G_5 + G_6G_4G_5 + G_6G_4G_5G_2H_1}{1 + (G_2H_1 + G_4H_2) + (G_2H_1\, G_4H_2)}$$

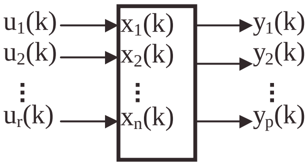


2.8 State Variables

Representing a System

transfer function

state variables



state vector

State Variable Method

i^{th} external input, $u_i(k)$ → *input space* = $\mathbf{u}(k)$
h^{th} system output, $y_h(k)$ → *output space* = $\mathbf{y}(k)$
j^{th} internal state variable, $x_j(k)$ → *state space* = $\mathbf{x}(k)$

- for any *given input* $u_i(k)$, this represents the minimal information needed to determine
  - *future states* $x_j(k+1)$
  - *system outputs* $y_j(k)$

ge**neral** description of non-linear, time varying system at time = (k+1): *(k implies kT)*

$$\mathbf{x}(k+1) = \mathbf{f}\,[\mathbf{x}(k),\mathbf{u}(k)] \qquad \textbf{(2-45)}$$

ge**neral** description of non-linear, time varying system response at time = (k)

$$\mathbf{y}(k) = \mathbf{g}\,[\mathbf{x}(k),\mathbf{u}(k)] \qquad \textbf{(2-46)}$$

description of linear system at time = (k+1):

$$\mathbf{x}(k+1) = \mathbf{A}(k)\mathbf{x}(k) + \mathbf{B}(k)\mathbf{u}(k) \qquad \textbf{(2-47)}$$

description of linear system response at time = (k)

$$\mathbf{y}(k) = \mathbf{C}(k)\mathbf{x}(k) + \mathbf{D}(k)\mathbf{u}(k) \qquad \textbf{(2-48)}$$

description of linear time invarient system at time = (k+1):

$$\mathbf{x}(k+1) = \mathbf{A}\ \mathbf{x}(k) + \mathbf{B}\ \mathbf{u}(k) \qquad \textbf{(2-49)}$$

description of linear time invarient system response at time = (k)

$$\mathbf{y}(k) = \mathbf{C}\ \mathbf{x}(k) + \mathbf{D}\ \mathbf{u}(k) \qquad \textbf{(2-50)}$$

ie:

$$y(k+2)=u(k)+1.7\,y(k+1)\text{-}0.72\,y(k)$$

$$x_1(k)=y(k)$$

$$x_2(k)=y_k(k+1)=x_1(k+1)$$

$$x_1(k+1)=x_2(k)$$

$$x_2(k+1)=u(k)+1.7\,x_2(k)\text{-}0.72\,x_1(k)$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.72 & 1.7 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k)=[1 \quad 0]\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Dervation of state-space equations from transfer function - control canonical form

(i) start with transfer function *G(z)*

$$G(z) = \frac{b_{n-1}z^{n-1} + b_{n-2}z^{n-2} + \ldots + b_0 z}{z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0{}^n} \tag{2-51}$$

for input *u(k)* & output *y(k)*:

$$\frac{Y(z)}{U(z)} = G(z) = \frac{b_{n-1}z^{n-1} + b_{n-2}z^{n-2} + \ldots + b_1 z + b_0}{z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0} \; \frac{E(z)}{E(z)} \tag{2-52}$$

$$Y(z) = (b_{n-1}z^{n-1} + b_{n-2}z^{n-2} + \ldots + b_0 z)\,E(z) \tag{2-53}$$

$$U(z) = (z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0)\,E(z) \tag{2-54}$$

(ii) From Real Translation property

$E(z) \rightarrow e(k)$

$zE(z) \rightarrow e(k+1)$

$z^n E(z) \rightarrow e(k+n) \ldots$

(iii) Define State Variables

$$\begin{aligned}
x_1(k) &= e(k) \\
x_2(k) &= x_1(k+1) = e(k+1) \\
x_3(k) &= x_2(k+1) = x_1(k+2) = e(k+2) \\
x_4(k) &= x_3(k+1) = x_2(k+2) = x_1(k+3) = e(k+3) \qquad (2\text{-}55) \\
&\ldots \\
x_n(k) &= x_{n-1}(k+1) = e(k+n-1)
\end{aligned}$$

(iv) obtain state equations - control canonical form

$$\begin{aligned}
x_1(k+1) &= x_2(k) \\
x_2(k+1) &= x_3(k) \\
&\ldots \\
x_n(k+1) &= -a_0\,x_1(k) - a_1\,x_2(k) - a_2\,x_3(k) - \ldots - a_{n-1}\,x_n(k) + u(k)
\end{aligned}$$

- (2-55) → $x_n(k+1) = e(k+n)$
- (2-54) → $u(k) = e(k+n) + a_{n-1}e(k+n-1) + \ldots + a_1\,e(k+1) + a_0 e(k)$
- $e(k+n) = u(k) - a_{n-1}e(k+n-1) - \ldots - a_1\,e(k+1) - a_0 e(k)$
- $x_n(k+1) = u(k) - a_{n-1}x_n(k) - \ldots - a_1\,x_2(k) - a_0 x_1(k)$

state-space equation

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ \dots \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0\dots0 \\ 0 & 0 & 1 & 0\dots0 \\ 0 & 0 & 0 & 1\dots0 \\ \dots & & & \\ -a_0 & -a_1 & \dots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ \dots \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dots \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} b_0 & b_1 & b_2 & b_3 \dots b_{n-1} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \dots \\ x_n(k) \end{bmatrix}$$

or $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$ **(2-59)**

$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k)$ **(2-60)**

→ eqn *(2-51)* = *transfer function representation*

→ eqn *(2-59)* & *(2-60)* = *state equations*

multiply (2-51) by z^{-n}

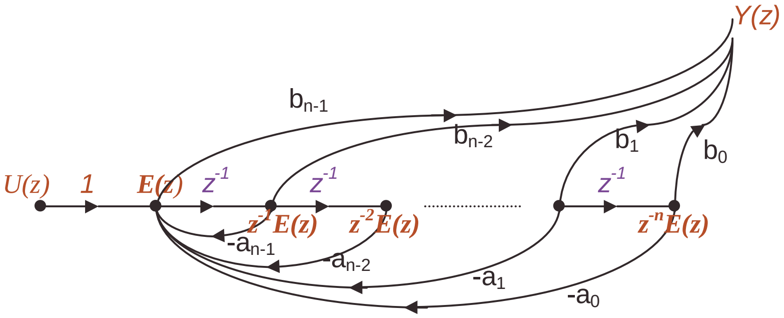
$$\frac{Y(z)}{U(z)} = G(z) = \frac{b_{n-1}z^{-1} + b_{n-2}z^{-2} + \dots + b_0 z^{-n}}{1 + a_{n-1}z^{-1} + \dots + a_1 z^{1-n} + a_0 z^{-n}} \frac{E(z)}{E(z)} \qquad \textbf{(2-61)}$$

$$Y(z) = (b_{n-1}z^{-1} + b_{n-2}z^{-2} + \dots + b_0 z^{-n})\, E(z) \qquad \textbf{(2-62)}$$

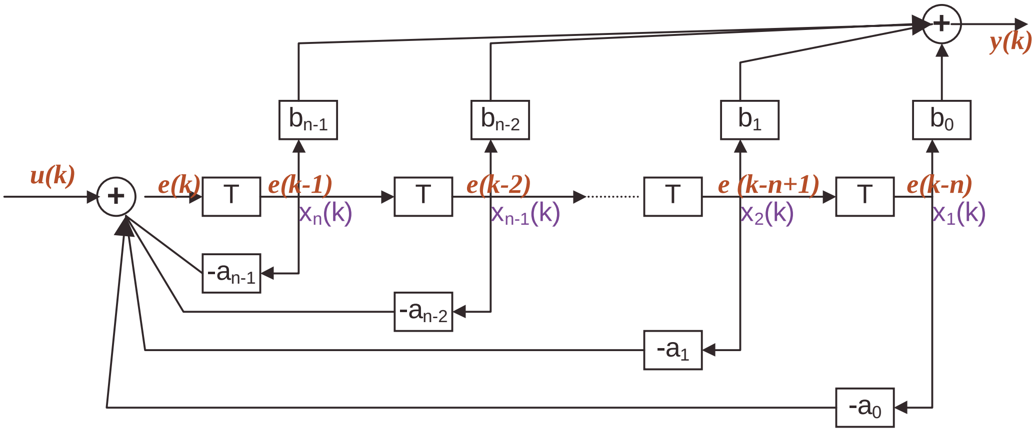
$$U(z) = (1 + a_{n-1}z^{-1} + \dots + a_1 z^{1-n} + a_0 z^{-n})\ E(z) \qquad \textbf{(2-63)}$$

$$E(z) = U(z) - (a_{n-1}z^{-1} + \dots + a_1 z^{1-n} + a_0 z^{-n})\, E(z) \qquad \textbf{(2-64)}$$

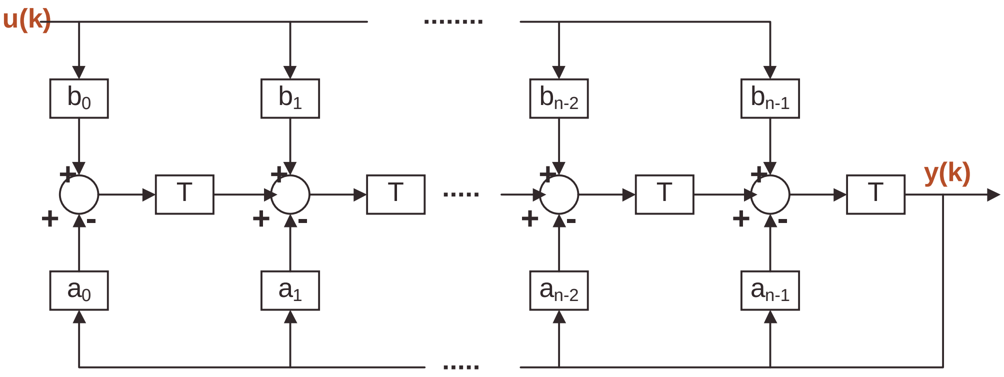
- derive **signal flow** graph for *control canonical form* from, use dummy variable, $E(z)$
  - $Y(z) \quad = \quad b_{n-1} z^{-1} E(z) + b_{n-2} z^{-2} E(z) + \ldots + b_0 z^{-n} E(z))$ (2-62’)
  - $E(z) \quad = \quad U(z) - (a_{n-1} z^{-1} E(z) + \ldots + a_1 z^{1-n} E(z) + a_0 z^{-n} E(z))$ (2-64’)
  - TF of delay of T seconds is z^{-1}



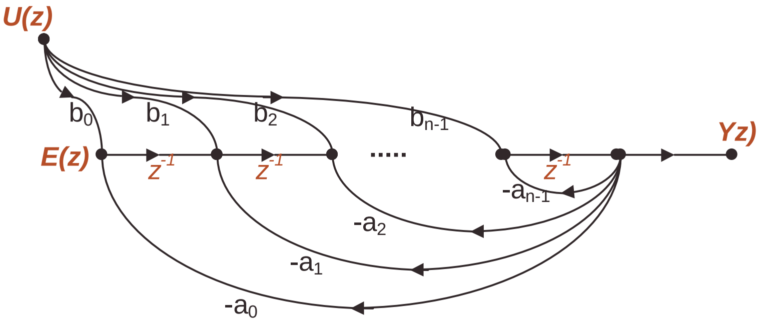
- derive **simulation diagram** for *control canonical form* from **signal flow**



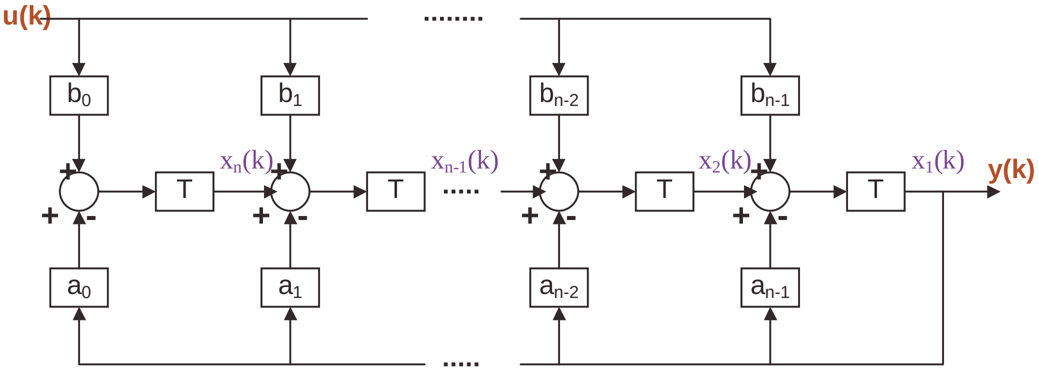
- **simulation diagram** for *observer canonical form* from **signal flow**



- **signal flow** graph for *observer canonical form* from



- derive **TF** for *observer canonical form*



$x_1(k+1) = x_2(k) - a_{n-1}x_1(k) + b_{n-1}u(k)$
$x_2(k+1) = x_3(k) - a_{n-2}x_1(k) + b_{n-2}u(k)$
…
$x_{n-1}(k+1)= x_n(k) - a_1x_1(k) + b_1u(k)$
$x_n(k+1) = - a_0x_1(k) + b_0u(k)$

and $y(k) = x_1(k)$

thus

$$\mathbf{x}(k+1) = \begin{bmatrix} -a_{n-1} & 1 & 0 & 0\ldots \\ -a_{n-2} & 0 & 1 & 0\ldots \\ & \cdots & & \\ -a_1 & 0 & 0\ldots & 1 \\ -a_0 & 0 & 0\ldots & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} b_{n-1} \\ b_{n-2} \\ \vdots \\ b_0 \end{bmatrix} u(k)$$

$y(k) = [1\ 0\ 0\ \ldots\ 0]\ \mathbf{x}(k)$

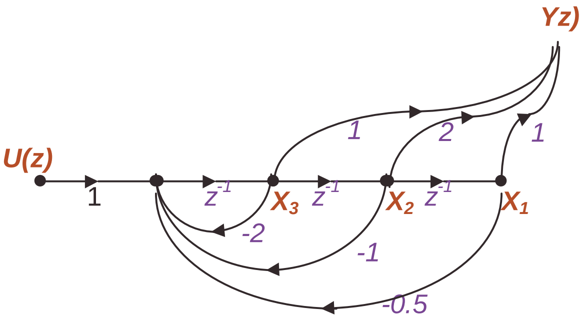
ie: control canonical form

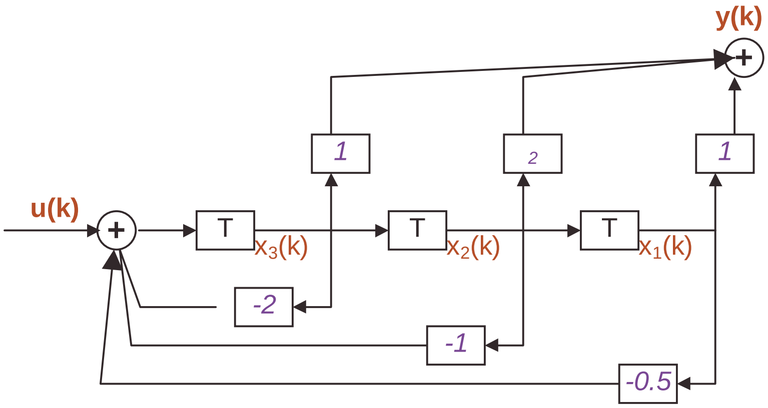
$$G(z)=\frac{Y(z)}{U(z)}=\frac{z^2+2z+1}{z^3+2z^2+z+\frac{1}{2}}$$

$$\begin{aligned}
x_1(k)&=e(k)\\
x_2(k)&=x_1(k+1)=e(k+1)\\
x_3(k)&=x_2(k+1)=e(k+2)\\
x_3(k+1)&=e(k+3)=u(k)-2x_3(k)-x_2(k)-0.5x_1(k)\\
y(k)&=x_3(k)+2x_2(k)+x_1(k)
\end{aligned}$$

$$\begin{bmatrix}x_1(k+1)\\x_2(k+1)\\x_3(k+1)\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\\-0.5&-1&-2\end{bmatrix}\begin{bmatrix}x_1(k)\\x_2(k)\\x_3(k)\end{bmatrix}+\begin{bmatrix}0\\0\\1\end{bmatrix}u(k)$$

$$y(k)=\begin{bmatrix}1&2&1\end{bmatrix}\begin{bmatrix}x_1(k)\\x_2(k)\\x_3(k)\end{bmatrix}$$

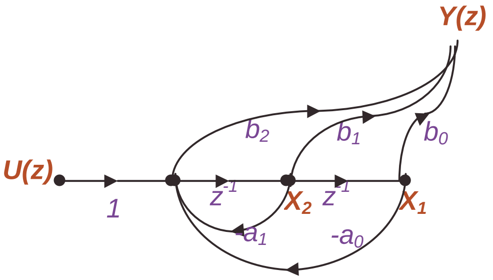




ie: numerator & denominator w/ same order → Direct path from Y(z) to E(z)

$$G(z) = \frac{Y(z)}{U(z)} = \frac{b_2 z^2 + b_1 z^1 + b_0}{z^2 + a_1 z + a_0\,)}$$

$$\frac{Y(z)}{U(z)} = \frac{b_2 + b_1 z^{-1} + b_0 z^{-2}\,E(z)}{1 + a_1 z^{-1} + a_0 z^{-2}\,E(z)}$$



$$x_1(k) = e(k-2)$$
$$x_2(k) = x_1(k+1) = e(k-1)$$
$$x_2(k+1) = u(k) - a_1\,x_2(k) - a_0\,x_1(k)$$

$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 \\ -a_o & -a_1 \end{bmatrix}\mathbf{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$y(k) = [b_0\ b_1\ b_2\ ]\mathbf{x}(k)$ *-wrong !, only 2 state variables*

from signal flow graph:

$$Y(z) = \mathbf{b_2 E(z)} + b_1 X_2(z) + b_0 X_1(z)$$
$$E(z) = U(z) - a_1 X_2(z) - a_0 X_1(z)$$

thus $Y(z) = b_2 U(z) + (b_1 - b_2 a_1) X_2(z) + (b_0 - b_2 a_0) X_1(z)$

and $y(k) = [(b_0 - b_2 a_0) \quad (b_1 - b_2 a_1)]\,\mathbf{x}(k) + b_2\,u(k)$

Deriving State Models:

(1) from transfer function:

$$G(z) = \frac{Y(z)E(z)}{U(z)E(z)}$$

$$x_i(k) = e(k-j)$$
$$x_i(k+1) = x_i(k)$$
$$\ldots$$

(2) from Simulation Diagram

- Assign state variable to each delay output
- Write equations for *delay input* & *system output* in terms of *delay output* & *system input*

(3) Decomposition of High Order TF's into product of Simpler TF's

Represent G(z) as product of simpler TF's: $G(z) = G_{1c}(z)G_{2c}(z)\ldots G_{nc}(z)$
Realize $G_{ic}(z)$ by either technique (1) or (2)
Connect $G_{ic}(z)$'s by cascading them
Use 2nd order $G_{ic}(z)$'s to avoid complex poles

(4) Decomposition of High Order TF's into sum of Simpler TF's

Represent G(z) as product of simpler TF's: $G(z) = G_{1p}(z) + G_{2p}(z) + \ldots + G_{np}(z)$
Realize $G_{ip}(z)$ by either technique (1) or (2)
Connect $G_{ic}(z)$'s in parallel

2.9 Other State-Variable Formulations

Transfer Function:

- gives system input/output relationship
- transfer function is unique for a given system

State Model:

- gives internal system description (not unique)
- gives input/output relationship
- derived from *transfer function, difference equation*, or *simulation diagram*

ie: $y(k+2) = u(k) + 1.7y(k+1) - 0.72y(k)$ - *derived earlier*

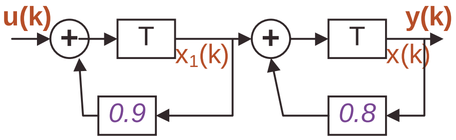
$$(1)\; x_1(k) = y(k)$$

$$x_2(k) = x_1(k+1) = y(k+1)$$

$$x_2(k+1) = u(k) + 1.7x_2(k) - 0.72x_1(k)$$

$$\bar{x}(k+1) = \begin{bmatrix} 0 & 1 \\ -0.72 & 1.7 \end{bmatrix}\bar{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(k), \qquad y(k) = [1 \;\; 0]\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

$$(2)\; \frac{Y(z)}{U(z)} = \frac{1}{z^2 - 1.7z + 0.72} = \left[\frac{1}{z-0.8}\right]\left[\frac{1}{z-0.9}\right]$$



start from simpler TF: $y(k) = x(k),\; x(k+1) = u(k) + 0.8x_1(k)$

$y_1(k) = x_1(k),\; x_1(k+1) = u_1(k) + 0.9x_1(k)$

overall state model:

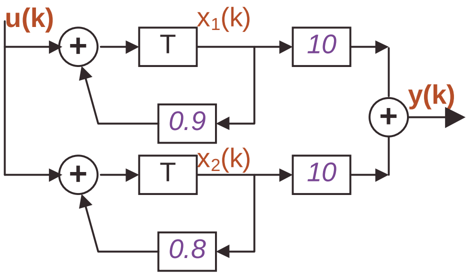
$$\bar{x}(k+1) = \begin{bmatrix} 0.8 & 1 \\ 0 & 0.9 \end{bmatrix}\bar{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(k), \qquad y(k) = [1 \;\; 0]\bar{x}(k)$$

$$(3)\; \frac{Y(z)}{U(z)} = \frac{1}{z^2 - 1.7z + 0.72} = \left[\frac{10}{z-0.9}\right] + \left[\frac{-10}{z-0.8}\right]$$

start from simpler TF: $y(k) = 10e(k),\; u(k) = e(k+1) - 0.8e(k)$

$y_1(k) = -10e(k),\; u_1(k) = e(k+1) - 0.9e(k)$

overall state model: $$\bar{x}(k+1) = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.9 \end{bmatrix}\bar{x}(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix}u(k), \qquad y(k) = [10 \;\; -10]\bar{x}(k)$$

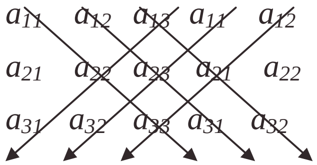


review of determinant and trace

(1) tr(A) = sum of elements on diagonal ($a_{11} + a_{22} + \ldots + a_{nn}$)

(2) $|A| = \sum(\pm)\, a_{1j_1}, a_{1j_2}, \ldots, a_{1j_n}$

- summation over all permutations $j_1, j_2, j_3, \ldots j_n$ of set S={1...n}
- '+' for even permutation, '-' for odd permutation
- $|A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$



Similarity Transformations of state model:

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k), \qquad (2\text{-}67)$$

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k)$$

- let $\mathbf{x}$, $\mathbf{w}$: *$1 \times n$ state vectors*
- let $\mathbf{P}$: $n \times n$ non-singular matrice → *each unique* **P** *yields a different state model*
- apply *linear transformation* to $\mathbf{x}(k)$ → $\mathbf{x}(k) = \mathbf{P}\mathbf{w}(k)$ (2-68)

$$x_1(k) = p_{11}w_1(k) + p_{12}w_2(k) + \ldots p_{1n}w_n(k)$$

$$x_2(k) = p_{21}w_2(k) + p_{22}w_2(k) + \ldots p_{2n}w_n(k)$$

$$\ldots$$

$$x_n(k) = p_{n1}w_1(k) + p_{n2}w_2(k) + \ldots p_{nn}w_n(k)$$

substitution of (2-68) into (2-67) yields:

$$\mathbf{w}(k+1) = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{w}(k) + \mathbf{P}^{-1}\mathbf{B}\mathbf{u}(k),$$

$$\mathbf{y}(k) = \mathbf{C}\mathbf{P}\mathbf{w}(k) + \mathbf{D}\mathbf{u}(k)$$

let $\mathbf{A}_w = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$
$\mathbf{B}_w = \mathbf{P}^{-1}\mathbf{B}$
$\mathbf{C}_w = \mathbf{C}\mathbf{P}$
$\mathbf{D}_w = \mathbf{D}$

then $\mathbf{w}(k+1) = \mathbf{A}_w\,\mathbf{w}(k) + \mathbf{B}_w\,\mathbf{u}(k)$

$\mathbf{y}(k) = \mathbf{C}_w\,\mathbf{w}(k) + \mathbf{D}_w\,\mathbf{u}(k)$

- **characteristic equation** of **A** ≡ determinant: $|z\mathbf{I}-\mathbf{A}|=0$

$$(z\mathbf{I}-\mathbf{A}) = \left\|\begin{bmatrix} z-a_{11} & -a_{12} \\ -a_{21} & z-a_{22} \end{bmatrix}\right\|$$

$$\begin{aligned} |z\mathbf{I}-\mathbf{A}| &= (z-a_{11})(z-a_{22}) - (a_{12}a_{21}) \\ &= (z-z_1)(z-z_2) \end{aligned}$$

- **characterisitc (*eigenvalues*) values:** roots of characteristic equation

  z_i = **eigenvalue** of **A** → $|z\mathbf{I}-\mathbf{A}| = (z-z_1)(z-z_2)\ldots(z-z_n)=0$

  $|z\mathbf{I}-\mathbf{A}|$ *determines stability for system*

  characteristic equation unchanged by lineat transformation: $|z\mathbf{I}-\mathbf{A}| = |z\mathbf{I}-\mathbf{A}_w|$

Similartiy transform*:* $\mathbf{A}_w = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$

(i) eigen values, z_i are unchanged
(ii) $|\mathbf{A}_w| = |\mathbf{P}^{-1}\mathbf{A}\mathbf{P}| = |\mathbf{P}^{-1}|\,|\mathbf{A}|\,|\mathbf{P}| = |\mathbf{A}|$
(iii) trace, $\mathrm{tr}\,\mathbf{A}_w = \mathrm{tr}\,\mathbf{A} = z_1 + z_2 + \ldots + z_n$
(iv) $\mathbf{C}\,|z\mathbf{I}-\mathbf{A}|^{-1}\,\mathbf{B} + \mathbf{D} = \mathbf{C}\,|z\mathbf{I}-\mathbf{A}_w|^{-1}\,\mathbf{B} + \mathbf{D}$

If a system has **distinct** *eigenvalues* → then derive a state variable model w/ diagonal system matrix

consider vector $\mathbf{m_i}$ and scaler z_i defined by: let $\mathbf{Am_i} = z_i\mathbf{m_i}$

where $\mathbf{m_i} = [m_{1i}, m_{2i, \ldots,} m_{ni}]^T$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} m_{1i} \\ m_{2i} \\ m_{3i} \end{bmatrix} = \begin{bmatrix} a_{11}m_{1i} + a_{12}m_{2i} + a_{13}m_{3i} \\ a_{21}m_{1i} + a_{22}m_{2i} + a_{23}m_{3i} \\ a_{31}m_{1i} + a_{32}m_{2i} + a_{33}m_{3i} \end{bmatrix} = \begin{bmatrix} zm_{1i} \\ zm_{2i} \\ zm_{31} \end{bmatrix}$$

- if $(z_i\mathbf{I} - \mathbf{A})\mathbf{m_i} = 0$ has a nontrivial solution → $|z_i\mathbf{I} - \mathbf{A}| = \mathbf{0}$ and z_i is eigenvalue of **A**

$$\begin{bmatrix} z - a_{11} & -a_{12} & -a_{13} \\ -a_{22} & z - a_{22} & -a_{23} \\ -a_{31} & -a_{32} & z - a_{33} \end{bmatrix} \begin{bmatrix} m_{11} \\ m_{21} \\ m_{31} \end{bmatrix} = \begin{bmatrix} (z - a_{11})m_{1i} - a_{12}m_{2i} - a_{13}m_{3i} \\ -a_{21}m_{1i} - (z - a_{22})m_{2i} + a_{23}m_{3i} \\ -a_{31}m_{1i} - a_{32}m_{2i} + (z - a_{33})m_{3i} \end{bmatrix} = 0$$

$\mathbf{m_i}$ = **eigenvector** of **A** and $\mathbf{A}[\mathbf{m_1}\ \mathbf{m_2} \ldots \mathbf{m_n}] = [\mathbf{m_1}\ \mathbf{m_2} \ldots \mathbf{m_n}]z\mathbf{I}$

$\mathbf{M} = [\mathbf{m_1}\ \mathbf{m_2} \ldots \mathbf{m_n}]$: **modal matrix**

Λ= **diaganol matrix** with **eigenvalues** of **A** where $\mathbf{\Lambda} = \mathbf{M^{-1}AM}$

ie: $\mathbf{x(k+1)} = \begin{bmatrix} 0.8 & 1 \\ 0 & 0.9 \end{bmatrix} \mathbf{x(k)} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$

$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x(k)}$

characteristic eqn: |zI-A| = **0** → $(z-0.8)(z-0.9) = 0$

eigenvalues:

$z_1 = 0.8$

$z_2 = 0.9$

then setting $\mathbf{Am_i} = z_i\mathbf{m_i}$ yields $\begin{bmatrix} 0.8 & 1 \\ 0 & 0.9 \end{bmatrix} \begin{bmatrix} m_{11} \\ m_{21} \end{bmatrix} = 0.8 \begin{bmatrix} m_{11} \\ m_{21} \end{bmatrix}$

$0.8m_{11} + m_{21} = 0.8m_{11}$

$0.9m_{21} = 0.8m_{21}$

thus m_{11} = arbitrary and $m_{21}=0$

for column two: $\begin{bmatrix} 0.8 & 1 \\ 0 & 0.9 \end{bmatrix} \begin{bmatrix} m_{12} \\ m_{22} \end{bmatrix} = 0.9 \begin{bmatrix} m_{12} \\ m_{22} \end{bmatrix}$

$0.8m_{12} + m_{22} = 0.9m_{12}$

$0.9m_{22} = 0.9m_{22}$

thus m_{22} arbitrary and $m_{12} = 10m_{22}$

and **eigenvectors** of A = $\begin{bmatrix} x \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 10y \\ y \end{bmatrix}$

if $x = 1$ and $y = 1$ → $\mathbf{M} = \begin{bmatrix} 1 & 10 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{M}^{-1} = \begin{bmatrix} 1 & -10 \\ 0 & 1 \end{bmatrix}$

$$\Lambda = \mathbf{M}^{-1}\mathbf{A}\mathbf{M} = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.9 \end{bmatrix}$$

$$\mathbf{B}_w = \mathbf{M}^{-1}\mathbf{B}\begin{bmatrix} -10 \\ 1 \end{bmatrix}$$

$$\mathbf{C}_w = \mathbf{C}\mathbf{M} = \begin{bmatrix} 1 & 10 \end{bmatrix}$$

and finally:

$$\mathbf{w(k+1)} = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.9 \end{bmatrix} \mathbf{w(k)} + \begin{bmatrix} -10 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 10 \end{bmatrix} \mathbf{w(k)}$$

2.10 Obtaining Transfer Function from State Model

given state space equations → construct *simulation diagram, signal flow graph*

(1) obtain difference equation relationship → take *z-transform*

(2) obtain transfer function from *reduction* or *Mason's* gain formula

(3) take *z-transform* of state equations & eliminate state varible, **x**

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) \quad \rightarrow z\mathbf{X}(z) - z\mathbf{x}(0) = \mathbf{A}\mathbf{X}(z) + \mathbf{B}\mathbf{U}(z)$$
$$\mathbf{y}(k) = Cx(k) + \mathbf{D}u(k) \quad \rightarrow \mathbf{Y}(z) = \mathbf{C}\mathbf{X}(z) + \mathbf{D}\mathbf{U}(z)$$

if initial condition = 0 →

$$[z\mathbf{I} - \mathbf{A}]\mathbf{X}(z) = \mathbf{B}\mathbf{U}(z)$$
$$\mathbf{X}(z) = [z\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\mathbf{U}(z)$$

then $\mathbf{Y}(z) = [\mathbf{C}\,[z\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D}]\,\mathbf{U}(z)$

and transfer function: $\mathbf{G(z)} = \mathbf{C}\,[z\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D}$

2.11 Solutions of State Equations

1. Recursive Solution - assume time invarient system w/ $\mathbf{x}(0)$ and *u(j) j=1,2,3…*known

$\mathbf{x}(k+1) = \mathbf{Ax}(k) + \mathbf{Bu}(k)$
$\mathbf{y}(k) = Cx(k) + \mathbf{D}u(k)$

$\mathbf{x}(1) = \mathbf{Ax}(0) + \mathbf{Bu}(0)$

$$\begin{aligned}\mathbf{x}(2) &= \mathbf{Ax}(1) + \mathbf{Bu}(1) \\ &= \mathbf{A}(\mathbf{Ax}(0) + \mathbf{Bu}(0)) + \mathbf{Bu}(1) \\ &= \mathbf{A}^2\mathbf{x}(0) + \mathbf{ABu}(0) + \mathbf{Bu}(1)\end{aligned}$$

$$\begin{aligned}\mathbf{x}(3) &= \mathbf{Ax}(2) + \mathbf{Bu}(2) \\ &= \mathbf{A}(\mathbf{A}^2\mathbf{x}(0) + \mathbf{ABu}(0)) + \mathbf{Bu}(1)) + \mathbf{Bu}(2) \\ &= \mathbf{A}^3\mathbf{x}(0) + \mathbf{A}^2\mathbf{Bu}(0) + \mathrm{A}\mathbf{Bu}(1) + \mathbf{Bu}(2)\end{aligned}$$

$$\mathbf{x}(k) = \mathbf{A}^k\mathbf{x}(0) + \sum_{i=0}^{k-1} \mathbf{A}^{k-1-i}\mathbf{B}u(i) \qquad (2\text{-}88)$$

let $\mathbf{\Phi}(k) = \mathbf{A}^k$ - *state transition matrix* or *fundamental matrix*

$$\mathbf{x}(k) = \mathbf{\Phi}(\mathbf{k})\, \mathbf{x}(0) + \sum_{i=0}^{k-1} \mathbf{\Phi}(k-1-i)\mathbf{B}u(i) \qquad (2\text{-}89)$$

$$y(k) = \mathbf{C\Phi}(k)\mathbf{x}(0) + \sum_{i=0}^{k-1} \mathbf{C\Phi}(k-1-i)\, \mathbf{B}u(i)$$

2. z-transform method:

$\mathbf{x}(k+1) = \mathbf{Ax}(k) + \mathbf{Bu}(k)$
$\mathbf{y}(k) = Cx(k) + \mathbf{D}u(k)$

let $u(k) = 0$ → $z\mathbf{X}(z) - z\mathbf{x}(0) = \mathbf{AX}(z)$
$(z\mathbf{I}-\mathbf{A})\mathbf{X}(z) - z\mathbf{x}(0)$
$\mathbf{X}(z) = z[zI-A]^{-1}\mathbf{x}(0)$
then $\mathbf{x}(k) = Z^{-1}\{\mathbf{X}(z)\} = Z^{-1}\{z[z\mathbf{I}-\mathbf{A}]^{-1}\mathbf{x}(0)\}$

if $u(k) = 0$ → $\mathbf{x}(k) = \mathbf{\Phi}(\mathbf{k})\, \mathbf{x}(0)$ from (2-89)
then $\mathbf{\Phi}(\mathbf{k})\, \mathbf{x}(0) = Z^{-1}\{z[z\mathbf{I}-\mathbf{A}]^{-1}\mathbf{x}(0)\}$
thus substitute $\mathbf{\Phi}(k) = Z^{-1}\{z|\, z\mathbf{I}-\mathbf{A}|^{-1}\}$

Properties of $\mathbf{\Phi}(k)$

- $\mathbf{x}(k) = \mathbf{\Phi}(k)\, \mathbf{x}(0)$ → $\mathbf{\Phi}(0) = \mathbf{I}$
- $\mathbf{\Phi}(k_1 + k_2) = A^{k_1+k_2} = A^{k_1}A^{k_2} = \mathbf{\Phi}(k_1)\mathbf{\Phi}(k_1)$
- $\mathbf{\Phi}(-k) = A^{-k} = [A^k]^{-1} = \mathbf{\Phi}^{-1}(k)$
- $\mathbf{\Phi}(k) = \mathbf{\Phi}^{-1}(-k)$

ie: $G(z) = \frac{(z+3)}{(z+1)(z+2)} = \frac{(z+3)}{(z^2+3z+2)} = \frac{Y(z)}{U(z)}$

$u(k) = e(k+2)+3e(k+1)+ 2e(k)$
$y(k) = e(k+1) + 3e(k)$
$x_1(k) = e(k)$
$x_2(k) = e(k+1)$
$x_2(k+1) = u(k) - 3x_2(k) - 2x_1(k)$

$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$
$$y(k) = \begin{bmatrix} 3 & 1 \end{bmatrix} x(k)$$

asssume

system is initially at rest: $\mathbf{x}(0) = 0$
input is unit step: $u(k) = 1$ for k = 0,1,2…

$$x(1) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x(0) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$y(1) = \begin{bmatrix} 3 & 1 \end{bmatrix} x(1) = 1$$

$$x(2) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x(1) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(1) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$
$$y(2) = \begin{bmatrix} 3 & 1 \end{bmatrix} x(2) = 1$$

$$x(3) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x(2) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(2) = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$
$$y(3) = \begin{bmatrix} 3 & 1 \end{bmatrix} x(2) = -1$$

…