Stability of the flow of incompressible viscous fluid in a circular pipe is studied numerically. A perturbation consisting of finite-amplitude two-dimensional and infinitesimal three-dimensional parts is imposed on the basic flow. The temporal evolution of the perturbation is analyzed by direct numerical calculation of the Navier–Stokes equations. The two-dimensional disturbances are independent of the streamwise coordinate and initially take the form of streamwise rolls. It is shown that the nonlinear development of two-dimensional perturbations results in substantial spanwise modulation of the streamwise velocity component manifesting itself as a formation of streaks and the occurrence of inflection points. The modulated mean flow is found to be highly unstable to the three-dimensional perturbations which are localized spatially near these points. An instability mechanism that includes the modulation of the flow by growing two-dimensional disturbances and the inflectional instability of the modulated flow to three-dimensional perturbations is proposed.

I. INTRODUCTION

The transition from a laminar flow to a turbulent one in a pipe of circular cross section is one of the most intriguing problems of classical hydrodynamics. In spite of the numerous experimental and theoretical investigations of the subject performed since the original work of Reynolds, an understanding of the mechanism of transition is far from complete.

Experiments have shown that there is a critical Reynolds number such that a sustained transition to turbulence is impossible for since all disturbances of the laminar flow will decay in this case. The value of can be placed in the range . At , a critical initial amplitude of disturbances must be exceeded for the transition to occur. This amplitude is shown to be a slowly decreasing function of (Ref. 6). In precise experiments with tightly controlled level of disturbances, a laminar flow was observed at . A short comment should be made concerning the importance of the inlet region of a pipe to transition. In several experimental and theoretical studies inlet disturbances were considered as a source of turbulent motions. After the experiments of Rubin and Darbyshire and Mullin, it seems to be established that the fully developed pipe flow with a parabolic velocity profile has an effective internal mechanism to sustain the development of perturbations.

The classical stability and bifurcation theory cannot explain the instability in pipe flow. Linear stability analysis (see Refs. 11–15) has shown that a parabolic profile is stable to axisymmetric perturbations and is very likely to be stable to three-dimensional ones. The disagreement between the experiment and linearized theory has been assigned to nonlinear effects. The use of traditional weakly nonlinear analysis is hampered by the absence of a neutral stability curve. However, singular neutral modes at high were identified by Smith and Bodony. In this work, the flow subjected to three-dimensional disturbances was modeled with a nonlinear critical layer.

In understanding the route to turbulence in viscous shear flows has been made in last few years in the theoretical studies by Boberg and Brosa, Bergström, Schmid and Henningson, Trefet et al., and others. Further references can be found in the paper by Waleffe. It has been shown that for shear flows, the operator of the linearized Navier-Stokes equations is non-normal. As a result, some solutions of the linear stability problem, which are initially superpositions of several eigenmodes, exhibit a substantial algebraic growth before they begin to decay exponentially. It was suggested that the transient amplification of linear disturbances could be sufficient to trigger nonlinear effects which in turn would prevent the viscous decay of the perturbations.

In the case of pipe flow, the maximum linear amplification was obtained for two-dimensional perturbations which are independent of the streamwise coordinate and have the azimuthal wave number . The initial form of the perturbations is that of streamwise rolls. After a short time, the streamwise velocity component becomes dominating. The perturbations take the form of spanwise modulation of the basic flow leading to the formation of streaks which were observed in other shear flows. As described in Sec. III of the present paper, finite-amplitude disturbances demonstrate a very similar behavior.

An indication that streamwise independent structures play an important part in the transition can be found in the paper by Nikitin. Nonlinear development of a random perturbation was calculated at and the streamwise wave number , the initial rms velocity of perturbation being 2% of the centerline velocity of basic flow. It was shown that the initial growth of the perturbation is essentially due to the growth of the streamwise-independent component. Also, experiments carried out by and Fredss for the plane Poiseuille flow provided support for the numerical investigations. Point-like disturbances were observed to exhibit a transient growth, the structures of the largest growth being highly elongated in the streamwise direction.

The attempts to explain the breakdown of laminar shear
flows by the amplification of infinitesimal disturbances were criticized in the recent paper of Waleffe.\textsuperscript{22} The main objection was that the streamwise-independent perturbations experiencing the largest transient growth cannot be self-sustaining. Their nonlinear development adjusts the mean flow in such a way that non-normality and thus a potential for transient growth are reduced. It can be proved using the energy integrals for the Navier–Stokes equations that the streamwise-independent finite-amplitude perturbations of parallel shear flows must eventually decay.

An almost apparent conclusion can be drawn from the preceding. If the two-dimensional streamwise-independent disturbances are not able to initiate the transition to turbulence, but we want to use their ability to grow and modulate the mean flow, we should add three-dimensionality. In other words, we can study the stability of the mean flow modulated by the growing finite-amplitude streamwise-independent disturbance to the three-dimensional perturbations. It can be made in the framework of the full nonlinear problem or by weakly nonlinear analysis. Another way is chosen in the present paper. The Navier–Stokes equations are linearized around the modulated two-dimensional mean flow, and the temporal development of infinitesimal three-dimensional perturbations of this flow is investigated numerically. A similar approach was used by Orszag and Patera.\textsuperscript{25} In that study, the modulating finite-amplitude perturbations were axisymmetric. It is shown below that such modulation provides much less potential for three-dimensional growth than the modulation used in the present work.

The paper is organized as follows. The stability problem is formulated and the numerical method is outlined in Sec. II. The temporal behavior of finite-amplitude streamwise-independent perturbations is studied in Sec. III. The stability of modulated mean flow to the infinitesimal three-dimensional disturbances is investigated in Sec. IV. Finally, Sec. V presents the discussion of the results obtained.

II. FORMULATION OF THE PROBLEM AND NUMERICAL METHODS

The flow of a viscous incompressible fluid in a pipe of circular cross section is considered. A standard cylindrical coordinate system \((r, \phi, z)\) is used. Here, \(r\), \(\phi\), and \(z\) are the radial, azimuthal, and streamwise coordinates, respectively. The unperturbed mean flow driven by the constant pressure gradient is assumed to be of the form:

\[
\mathbf{W} = [0, 0, W(r)], \quad W = 1 - r^2, \quad P = 4(P_0 - z/\text{Re}),
\]

where the pipe radius \(R\) and the centerline velocity \(W_{\text{CL}}\) have been used for normalization, and \(P_0\) is a given constant. The Reynolds number is defined as \(\text{Re} = W_{\text{CL}}R/\nu\), where \(\nu\) is the kinematic viscosity of the fluid.

A perturbation consisting of two parts is added to the basic flow (1). First, a finite-amplitude disturbance depending on \(r\), \(\phi\), and time \(t\) is applied to modulate the basic flow. Second, an infinitesimal three-dimensional disturbance is superposed on the nonsteady modulated flow. The perturbed velocity field is

\[
\mathbf{V} = \mathbf{W} + \mathbf{V}^{(2)}(r, \phi, z, t) + \epsilon \mathbf{V}^{(3)}(r, \phi, z, t) \quad (\epsilon \ll 1),
\]

where \(\mathbf{V}^{(2)}\) is the finite-amplitude two-dimensional perturbation and \(\epsilon \mathbf{V}^{(3)}\) is the infinitesimal three-dimensional perturbation. Substituting (2) into the Navier–Stokes equations and separating the terms of zero and first order in \(\epsilon\) (second-order terms are neglected) one obtains nonlinear equations for \(\mathbf{V}^{(2)}\)

\[
\frac{\partial \mathbf{V}^{(2)}}{\partial t} + (\mathbf{V}^{(2)}, \nabla) \mathbf{W} + (\nabla, \nabla) \mathbf{V}^{(2)} = -\nabla p^{(2)} + \frac{1}{\text{Re}} \Delta \mathbf{V}^{(2)},
\]

[the term \((\mathbf{W}, \nabla) \mathbf{V}^{(2)}\) is dropped due to \(z\)-independence of \(\mathbf{V}^{(2)}\)]

\[
\text{div} \mathbf{V}^{(2)} = 0,
\]

and linearized equations for \(\mathbf{V}^{(3)}\)

\[
\frac{\partial \mathbf{V}^{(3)}}{\partial t} + (\mathbf{W}, \nabla) \mathbf{V}^{(3)} + (\mathbf{V}^{(2)}, \nabla) \mathbf{V}^{(3)} + (\nabla, \nabla) \mathbf{V}^{(3)} = -\nabla p^{(3)} + \frac{1}{\text{Re}} \Delta \mathbf{V}^{(3)},
\]

\[
\text{div} \mathbf{V}^{(3)} = 0,
\]

where \(p^{(2)}\) and \(\epsilon p^{(3)}\) are, respectively, two-dimensional and three-dimensional perturbations of the pressure field. The no-slip boundary conditions at the pipe wall read

\[
\mathbf{V}^{(2)} = \mathbf{V}^{(3)} = 0 \quad \text{at} \quad r = 1.
\]

At the centerline \(r = 0\) the conditions (see Ref. 26)

\[
\lim_{r \to 0} \frac{\partial \mathbf{V}}{\partial r} = \lim_{r \to 0} \frac{\partial P}{\partial r} = 0
\]

are imposed. Here, a pair \((\mathbf{V}, P)\) stands for \((\mathbf{V}^{(2)}, p^{(2)})\) or \((\mathbf{V}^{(3)}, p^{(3)})\). Conditions (8) ensure the boundedness and smoothness of the solution at \(r \to 0\) when the azimuthal wave number \(m\) (see below) satisfies the inequality \(|m| > 1\). In the cases \(m = 0\) and \(m = \pm 1\), additional conditions are required. They are deduced by enforcing the continuity equation and one of the momentum equations on the centerline.

The direct time-dependent solution of the initial-boundary value problem consisting of (3)–(8) and the initial conditions to be defined below is calculated by a combined, finite-difference in \(r\) and pseudo-spectral in \(\phi\) numerical method. The perturbation velocity and pressure fields are assumed to be

\[
(\mathbf{V}^{(2)}, p^{(2)}) = \sum_{m=-M/2}^{M/2-1} (\mathbf{V}^{(2)}_m, p^{(2)}_m)(r,t)e^{im\phi},
\]

\[
(\mathbf{V}^{(3)}, p^{(3)}) = \sum_{n=-1}^{M/2-1} \sum_{m=-M/2}^{M/2-1} (\mathbf{V}^{(3)}_{nm}, p^{(3)}_{nm})(r,t)e^{	ext{in} \alpha z}e^{im\phi}.
\]

When constructing the expansion (10), only one mode with wave number \(\alpha\) is kept in streamwise direction, since the equations (5) and (6) for three-dimensional perturbation are linear in \(\mathbf{V}^{(3)}\) and have \(z\)-independent coefficients. On the contrary, \(M\) modes should be kept in azimuthal direction due
to nonlinear coupling between $\mathbf{V}^{(2)}$ and $\mathbf{V}^{(3)}$. Because the fields on the left-hand sides of (9), (10) must be real, the following conditions are required:

$$
\mathbf{V}^{(2)}_{m} = \overline{\mathbf{V}}^{(2)}_{m}, \quad p^{(2)}_{m} = \overline{p}^{(2)}_{m}, \quad \mathbf{V}^{(3)}_{n-m} = \overline{\mathbf{V}}^{(3)}_{nm},
$$

$$
p^{(3)}_{n-m} = \overline{p}^{(3)}_{nm},
$$

where an overbar denotes a complex conjugate. Also, the coefficients with subscript $m = -M/2$ are set equal to zero. This allows one to reduce the computations. Only the coefficients $\mathbf{V}^{(2)}_{m}, p^{(2)}_{m}$ with $m \geq 0$ and $\mathbf{V}^{(3)}_{nm}, p^{(3)}_{nm}$ with $n = 1$ are calculated.

After substitution of the expansions (9), (10) into (3)–(6), the equations are discretized in $r$ using the central differences of the second order. The velocity fields $\mathbf{V}^{(2)}, \mathbf{V}^{(3)}$ are computed at the integer discretization points $r_j = j/(K+1)$, $(1 \leq j \leq K)$ and the pressure fields $p^{(2)}$ and $p^{(3)}$ are calculated at the half-integer points $r_{j-1/2} = (j-1/2)/(K+1), (1 \leq j \leq K+1)$. Momentum equations (3) and (5) are approximated at the integer points, and incompressibility equations (4) and (6) are approximated at the half-integer ones. The temporal discretization procedure is a combined implicit–explicit one. The nonlinear term in (3) and pseudononlinear terms in (5) (the third and fifth terms on the left-hand side) are treated using the 3/2-rule. Then the boundary-value difference problems in radial direction are solved by a simple and convenient variant of Gauss elimination known as the “double sweep method.”

During the calculations, the distribution of the energy of perturbations over the azimuthal wave numbers and over the velocity components was computed. For two-dimensional ($E^{(2)}$) and three-dimensional ($E^{(3)}$) perturbations, the total energy related to the energy of the basic flow (1) can be given as

$$
E^{(2)} = E^{(2)}_u + E^{(2)}_v + E^{(2)}_w
$$

$$
= \sum_{m=0}^{M/2-1} b_m \int_0^1 \left[|u_m^{(2)}|^2 + |v_m^{(2)}|^2 + |w_m^{(2)}|^2\right] r \, dr,
$$

(11)

$$
E^{(3)} = E^{(3)}_u + E^{(3)}_v + E^{(3)}_w
$$

$$
= \sum_{m=-M/2}^{M/2-1} b_m \int_0^1 \left[|u_{1m}^{(3)}|^2 + |v_{1m}^{(3)}|^2 + |w_{1m}^{(3)}|^2\right] r \, dr,
$$

(12)

where $b_m = 6$ if $m = 0$, and $b_m = 12$ if $|m| \geq 1$.

When performing most of the calculations, we used the following values of the discretization parameters. The number of trial functions in azimuthal direction $M = 32$, the number of discretization points in radial direction $K = 50$, and the time step $\Delta t = 0.01$. To test the effect of the discretization on the accuracy, control runs were performed with $M = 64$, $K = 100$, and $\Delta t = 0.005$. The results of calculations concerning the time dependence of the energy and the spatial form of perturbations were found to be insensitive to such change of the discretization parameters.

III. TWO-DIMENSIONAL MODULATION OF THE BASIC FLOW

To examine the modulation of the basic flow (1) by the finite-amplitude, streamwise-independent perturbation $\mathbf{V}^{(2)}$, the boundary-value problems (3), (4), (7), and (8) must be complemented by initial conditions. We use an initial perturbation velocity field in the form of streamwise rolls which are shown by previous calculations18–20 to exhibit the largest transient growth in the case of infinitesimal perturbations. At $t = 0$ the streamwise velocity component is zero and the perturbation has azimuthal periodicity with the wave number $m = 1$. The radial dependence of the velocity is chosen to be the simplest one satisfying the boundary and incompressibility conditions. If we denote the radial, azimuthal, and streamwise components as $u$, $v$, and $w$, respectively, the initial form of two-dimensional perturbation is

$$
\mathbf{V}^{(2)}_m = (u^{(2)}_m, v^{(2)}_m, w^{(2)}_m)
$$

$$
\begin{cases}
A^{(2)}(f_1(r), imf_2(r), 0) & \text{at } |m| = 1, \\
(0,0,0) & \text{at } |m| \neq 1,
\end{cases}
$$

(13)

where

$$
f_1(r) = 1 - 3r^2 + 2r^3, \quad f_2(r) = 1 - 9r^2 + 8r^3,
$$

(14)

and $A^{(2)}$ is a constant used to adjust the initial energy to a desirable value. According to (13) and (14) radial and azimuthal velocity components contain, respectively, 40% and 60% of the initial energy. Such a distribution coincides with that obtained by Bergström19 for the optimal (providing largest transient amplification) linear perturbation at Re = 1000.

As the result of calculations, Fig. 1 shows the time dependence of the energy $E^{(2)}(t)$ of two-dimensional perturbation.
tions related to the initial energy \( E^{(2)}(0) \). To compare nonlinear and linear behaviors, the solution of the linearized perturbation equations with the same initial conditions (13) and (14) was calculated (the dashed curve in Fig. 1). This linear solution can be compared with the linear solutions obtained by Shmid and Henningson.\(^{20}\) They used initial perturbations constructed as linear combinations of eigenmodes of the linear stability problem. “Optimal” initial conditions providing the maximum amplification of the perturbation energy at a given time \( t_0 \) were found. It was shown that at \( Re=3000 \) an envelope curve of maximum possible amplification at any \( t \) is approached very closely by the curve obtained with \( t_0=147 \) and having the maximum amplification factor \( E^{(2)}(t_0)/E^{(2)}(0)=649 \). For comparison, our calculations give maximum amplification \( E^{(2)}(t_{\text{max}})/E^{(2)}(0)=648.1 \) at \( t_{\text{max}}=146.9 \) if the number of radial discretization points \( N=50 \) and \( E^{(2)}(t_{\text{max}})/E^{(2)}(0)=648.4 \) at \( t_{\text{max}}=146.8 \) if \( N=100 \). We can, thus, conclude that the simple initial conditions (13) and (14) are very close to the optimal ones providing maximum potential for the transient growth of infinitesimal two-dimensional perturbations.

It can be seen in Fig. 1 that the development of finite-amplitude perturbations can be divided into two stages. During the first stage, at relatively small \( t \), the nonlinear curves follow the linear curve. The growth of the energy of finite-amplitude perturbations is almost solely due to this stage of “pseudolinear” growth. For the perturbations with smaller initial energy this stage is longer and, thus, they demonstrate larger amplification. The solutions with initial amplitude between 0.001 and 0.01 have comparable amplitudes at the times of maximum growth. The second stage, a very long one, is the stage of nonlinear development and is characterized by very slow decay of the perturbations.

A typical distribution of the total energy of two-dimensional finite-amplitude disturbance over different velocity components is shown in Fig. 2. It can be seen that the amplification is related solely to the streamwise component, with the radial and azimuthal components decaying very rapidly. Initial streamwise rolls disappear, giving rise to a span-

![FIG. 2. The distribution of the total energy of two-dimensional finite-amplitude perturbation over the velocity components when Re=3000, \( E^{(2)}(0)=10^{-2}, \ldots, E^{(2)}_r, \ldots, E^{(2)}_z, \ldots \), total energy \( E^{(2)} \).](image)

wise modulation of the basic velocity profile.

One can see in Fig. 3 that the degree and kind of the modulation depend strongly on the initial energy of the perturbation. If the energy is high, two symmetrically located regions appear in the cross section of pipe where the streamwise flow is stronger than in the surrounding fluid. Such structures were observed in other shear flows and are called sometimes streaks. An important feature of the modulated velocity profile shown in Fig. 3(a) is that unlike the unperturbed parabolic profile it has inflection points. As discussed below, streaks play a crucial role in the three-dimensional instability of modulated flow.

If the initial disturbance is weak, we still have a significant modification of the basic flow but streaks are less pronounced or do not appear at all [see Fig. 3(b)]. In this case, the development of two-dimensional perturbations results mainly in the formation of the region of larger velocity gradient near the pipe wall.

The last point to be discussed in this section is the influence of the Reynolds number on the amplification of two-dimensional perturbations. Figure 4 presents maximum energy \( E^{(2)}(t_{\text{max}}) \) as a function of the Reynolds number and the
initial amplitude $E^{(2)}(0)$. Although the amplification decreases with decreasing Reynolds number, a significant growth occurs even at the lowest Reynolds number $Re = 1000$. Therefore, we cannot explain the critical Reynolds number $Re_c \approx 2000$ observed in the experiments by the lack of transient growth of two-dimensional perturbations at smaller $Re$. Three-dimensional disturbances of the modulated mean flow should be considered.

As discussed in Sec. IV, the modulation in the form of streaks is of fundamental importance for three-dimensional instability. Because of this, the spatial structure of the modulated flow was checked at $t=100$ in each run. If $E^{(2)}(0) = 10^{-2}$, streaks are clearly visible at $Re \approx 2000$. At $Re = 1500$ they are weak and at $Re = 1000$ do not appear at all. Thus we can set the lower boundary for the development of streaks to be $1000 < Re < 2000$. If $E^{(2)}(0) = 10^{-3}$, a similar boundary can be defined as being between 1500 and 2000. In the case of $E^{(2)}(0) = 10^{-4}$, weak streaks were detected at $Re = 4000$. At the other Reynolds numbers the structure of the modulated mean flow is similar to that shown in Fig. 3(b), with the flow being more similar to unperturbed basic flow at smaller $Re$.

IV. THREE-DIMENSIONAL INSTABILITY OF THE MODULATED MEAN FLOW

The evolution of infinitesimal three-dimensional disturbances of the modulated mean flow is considered in this section. As in the two-dimensional case, Eqs. (5) and (6) and boundary conditions (7) and (8) should be complemented by initial conditions for the perturbation velocity field. Once again, we construct the simplest form of the initial velocity satisfying the boundary and incompressibility conditions. The coefficients $V_{nm}^{(3)}$ with $|m| > M/6$ are assumed to be zero. For the other coefficients we can write

$$V_{nm}^{(3)} = (u_{nm}^{(3)}, v_{nm}^{(3)}, w_{nm}^{(3)})$$

where

$$f_1(r) = r - 2r^2 + r^3,$$
$$f_2(r) = \frac{i}{m} \left[ 2r - (6 + n\alpha)r^2 + (4 + n\alpha)r^3 \right],$$
$$f_3(r) = i(r - r^2),$$
$$f_4(r) = 1 - 3r^2 + 2r^3,$$
$$f_5(r) = \frac{i}{m} \left[ 1 - (9 + n\alpha)r^2 + (8 + n\alpha)r^3 \right],$$
$$f_6(r) = n\alpha(2r - \frac{3}{2}r^3 + r^4),$$
$$f_7(r) = i(1 - 6r^2 + 5r^3),$$

and $\alpha$ is the streamwise wave number. The constants $A_m^{(3)}$ are selected in such a way as to provide homogeneous distribution of the perturbation energy over the azimuthal wave numbers $m$. Since the linearized equations are solved for three-dimensional perturbations, the initial total energy $E^{(3)}(0)$ can be chosen arbitrarily. We use $E^{(3)}(0) = 10^{-6}$ for all runs.

Three parameters define the development of three-dimensional disturbances imposed on the modulated mean flow. They are the Reynolds number $Re$, the streamwise wave number $\alpha$, and the initial energy of the modulation (measured in this study with respect to that of the basic flow) $E^{(2)}(0)$. The results of the calculations with $Re = 3000$, $\alpha = 1.0$, and $E^{(2)}(0) = 10^{-2}$ are presented in Fig. 5. The total energy of two- and three-dimensional perturbations is given as a function of time. For comparison, two more solutions...
also shown in Fig. 5 were obtained with the same parameters. One of them corresponds to the development of three-dimensional infinitesimal perturbations imposed on the unperturbed basic flow $W(r)$, the initial conditions being identical to those given by (15) and (16). The other solution was calculated to give a possibility to compare our results with those of Orszag and Patera.\textsuperscript{25} In this case, the two-dimensional perturbation is axisymmetric and $z$-dependent with the streamwise wave number $\alpha=1.0$. Three-dimensional perturbation has the same streamwise periodicity and the azimuthal wave number $m=1$. The initial velocity field is constructed in the same manner as in our calculations [see (13)–(16)]. It can be seen in Fig. 5 that the mean flow modulated by streamwise-independent perturbations used in the present study is much more unstable to three-dimensional disturbances than the unmodulated basic flow or the mean flow modulated by axisymmetric perturbations.

The spatial structure of a developed three-dimensional perturbation is presented in Fig. 6 for the case Re=3000, $\alpha=1.0$, and $E^{(2)}(0)=10^{-2}$. Contours of constant energy density $E^{(3)}$ integrated over the period in $z$ and taken at $t=150$ are shown. One can see that the perturbation is localized asymmetrically in two regions. When comparing Fig. 6 with Fig. 3(a), these regions can be related to the inflection points of modulated mean flow. To make this fact more obvious, Fig. 7(a) shows the streamwise velocity of the modulated flow in the vicinity of the lower point of maximum $E^{(3)}$ [denoted $(r^*,\phi^*)$]. One more illustration is given in Fig. 7(b). The abscissa is the distance $s$ from $(r^*,\phi^*)$ along the straight line connecting this point with the point of maximum streamwise velocity of modulated mean flow. The second derivative $\partial^2(W+w^{(2)})/\partial s^2$ in the direction of this line is plotted as a function of $s$ on the vertical axis. The point of zero second derivative can be considered as a counterpart of the inflection points of axisymmetric and plane shear flows. It can be seen that this point is very close to the point of maximum energy of three-dimensional perturbation $(r^*,\phi^*)$ [corresponding to $s=0$ in Fig. 7(b)].

![Image](a)

The spatial localization of growing three-dimensional perturbation near the inflection points of the mean flow was obtained with other parameters and can be considered as a common phenomenon for the case of the modulation used in the present study. The inflection points are far from the viscosity-dominated wall region. The inviscid instability mechanism, namely the one of the inflectional instability, seems to be operative. An additional supporting argument can be given if we calculate the speed of streamwise propagation of the three-dimensional wave at inflection point. It was made for the flow shown in Figs. 3(a) and 6, i.e., with Re=3000, $E^{(2)}(0)=10^{-2}$, $\alpha=1.0$, and $t=150$, the result being 0.602. This value is in rough agreement with the stream-
The energy of three-dimensional perturbations versus time is shown in Fig. 10 for Re=3000 and different α and E(2)(0). When E(2)(0)=10^{-2} [Fig. 10(a)], the most significant growth is for the perturbation with the streamwise wave number α=1.5. At t=200 the amplification E(3)(200)/E(3)(0) is 3.52×10^8. Clearly, in this case, even the perturbation, which is very weak initially, reaches finite amplitude in a short time. Calculations have shown that the perturbations with α between 1.0 and 2.0 experience approximately equal growth. They have spatial structures similar to that shown for α=1.0 in Figs. 6–9, i.e., the energy is localized asymmetrically in two regions near the inflection points of the mean flow.

In the case E(2)(0)=10^{-3} [Fig. 10(b)], the growth of three-dimensional perturbations is not as impressive as in Fig. 10(a) but is still significant. The preferred mode α=0.3 has a maximum amplification E(3)(147)/E(3)(0)=95.3. The spatial structure of this mode as well as of those with 0.1≤α≤1.0 is essentially the same as of the most rapidly growing perturbation at E(2)(0)=10^{-2}. The mechanism of the inflectional instability is operative. It can also be noted that the perturbations grow for shorter times than at E(2)(0)=10^{-2}. This fact cannot be related to more rapid decay of two-dimensional disturbances at E(2)(0)=10^{-3}. It is shown in Fig. 1 that this decay is fairly slow, the modulation still being significant at t=200.

There is a fundamental difference between the results obtained with E(2)(0)=10^{-4} [Fig. 10(c)] and those shown in Figs. 10(a) and 10(b). The most rapidly growing perturbations have vanishing streamwise wave numbers. The spatial structure of these modes is also of radically different kind from that of the preferred modes at E(2)(0)=10^{-2},10^{-3}. The perturbation energy is localized in the region near the pipe wall where mean flow has the largest velocity gradient. The energy curves for modes with small α practically coincide with the curve for two-dimensional infinitesimal disturbance with α=0. Obviously mode α=0, which has the same symmetry as finite-amplitude modulating perturbation, cannot
trigger instability. It seems reasonable to suppose that the growth of perturbations with vanishing wave numbers cannot be the reason for a transition to turbulence.

It can be seen in Fig. 10 that mode $\alpha=0.3$ experiences a transient growth, even if rather weak [maximum amplification $E^{(3)}(55)/E^{(3)}(0)=20.7$]. The spatial structure of this mode is shown in Fig. 11. The perturbation energy is localized in two regions. One of them, near the pipe wall, is typical for the perturbations with vanishing $\alpha$. The second one can be related to the inflection point of the mean flow which has very weak streaks [see Fig. 3(b)]. Therefore, a kind of inflectional instability mechanism is operative in this case.

To examine the influence of the Reynolds number of the three-dimensional instability of modulated mean flow, the calculations were performed in the following manner. The time evolution of three-dimensional perturbations imposed at $t=30$ was calculated for three different values of the initial energy of two-dimensional perturbation $E^{(2)}(0)=10^{-2}, 10^{-3}, 10^{-4}$ and for the Reynolds numbers in the range between 1000 and 4000. The streamwise wave number $\alpha$ was taken to be 1.5 at $E^{(2)}(0)=10^{-2}$ and 0.3 at $E^{(2)}(0)=10^{-3}, 10^{-4}$. The results are shown in Figs. 12(a)–12(c). It can be seen that the amplification of three-dimensional perturbations always increases monotonically with the Reynolds number.

At $E^{(2)}(0)=10^{-2}$ [Fig. 12(a)] significant growth of three-dimensional perturbations for fairly long times occurs at $Re\approx2000$. It is difficult to define the critical amplification of initially infinitesimal perturbation necessary to trigger non-linear mechanisms and, probably, to give rise to transition to turbulence. For example, we can adopt the following criterion (having little reasons for it). The mean flow is called unstable to three-dimensional perturbation if the amplification of this perturbation is larger than 50 during a period of time $\Delta t\geq 50$. One can see in Fig. 12(a) that under this assumption, the modulated flow with $E^{(2)}(0)=10^{-2}$ is unstable to mode $\alpha=1.5$ at $Re\approx2000$ and stable to this mode at $Re\approx1500$. The initial two-dimensional disturbance is very strong in this case (rms velocity of the disturbance is about 10% of that of the basic flow). Therefore, we can compare the critical Reynolds number 1500<Re<2000 obtained in the calculations with the experimental $Re_\text{c}$ below which all possible disturbances of the basic flow decay. The lowest
value \( \text{Re}_c = 1760 \) found in experiments\(^6\) is in agreement with our results.

When \( E^{(2)}(0) = 10^{-3} \) [see Fig. 12(b)], the growth of three-dimensional perturbations is not as large as in the case of \( E^{(2)}(0) = 10^{-2} \) but is still significant at \( \text{Re} \geq 3000 \). Applying the criterion stated above we can define ranges of stability and instability of the mean flow to the perturbation with \( \alpha = 0.3 \). It can be seen in Fig. 12(b) that the flow is stable at \( \text{Re} \leq 2500 \) and unstable at \( \text{Re} \geq 3000 \), the critical Reynolds number being slightly below 3000.

Figure 12(c) presents the results of the calculations with small initial energy of two-dimensional disturbance, \( E^{(2)}(0) = 10^{-4} \). Even at \( \text{Re} = 4000 \) the amplification of three-dimensional perturbation does not exceed 30. After short time of growth, perturbations begin to decay rapidly. None of the curves shown in Fig. 12(c) satisfies the instability criterion adopted. We can, thus, state that the modulated mean flow with \( E^{(2)}(0) = 10^{-4} \) is stable to the three-dimensional perturbations with \( \alpha = 0.3 \) at \( \text{Re} \leq 4000 \).

**V. DISCUSSION**

An attempt is made in the present work to find an explanation of transition to turbulence in the Hagen–Poiseuille flow. The time evolution of the infinitesimal three-dimensional perturbations imposed on the mean flow modulated by the finite-amplitude streamwise-independent disturbances is examined numerically. The two-dimensional disturbances are initially streamwise rolls. They eventually decay but before that, they grow and change the structure of mean flow. If two-dimensional disturbance has a large enough initial energy, its development results in spanwise modulation of the streamwise velocity of mean flow. The structures called streaks are generated and inflection points appear in the velocity profile [see Fig. 3(a)]. Mean flow modulated in such a way is found to be highly unstable to three-dimensional perturbations localized near the inflection points. The term “instability” is used here with the meaning that three-dimensional perturbations can grow considerably before the decay of two-dimensional disturbances causes them to decay.

The following possible scenario can be proposed for transition to turbulence in pipe flow. The development of the streamwise-independent part of arbitrary disturbance gives rise to the inflection points in the velocity profile. Then, the mechanism of inflectional instability becomes operative causing growth of the three-dimensional part of the disturbance.

This transition scenario seems to be common to a number of shear flows. For example, an extensive numerical study of streak breakdown was performed recently by Reddy, Schmid, and Baggett.\(^{28}\) Linear and nonlinear evolution of three-dimensional perturbations of streak velocity fields was investigated. It was shown that streaks are highly unstable and that the instability is the inflectional one.

Clearly the highly simplified explanation proposed above cannot explain all the features of pipe flow instability. The process of streak breakdown must be investigated in the framework of the full nonlinear problem with particular attention given to the ability of weak streamwise-dependent structures to sustain the streak-like modulation. A probable mechanism includes the maintenance of streamwise rolls by nonlinear self-interaction of the streamwise-dependent part.
of the velocity field. Then, streaks can be sustained via the evolution of the rolls.

To check the validity of the instability mechanism suggested in the present paper, experiments are also required. The experiments may include the observations of the velocity field developing after the introduction of the perturbations of prescribed form. Streamwise elongated structures similar to those observed in the plane Poiseuille flow24 would be of particular interest.

ACKNOWLEDGMENTS

The author would like to thank Professor Karl G. Roesner for very useful and stimulating discussions and Dr. Nikolay V. Nikitin for constructive comments on the paper. The support provided by the Alexander von Humboldt Foundation is greatly acknowledged.

1 O. Reynolds, “An experimental investigation of the circumstances which determine whether the motion of water shall be direct or sinuous and of the law of resistance in parallel channels,” Proc. R. Soc. London 35, 84 (1883).