Statistics of turbulence in a generalized random-force-driven Burgers equation

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The statistics of solutions to a family of one-dimensional random-force-driven advection-diffusion equations is studied using high resolution numerical simulations. The equation differs from the usual Burgers equation by the non-local form of the nonlinear interaction term mimicking the non-locality of the Navier–Stokes equation. It is shown that under an appropriate choice of random forcing the statistical properties of the solution (energy spectrum and scaling exponents of structure functions) coincide with those of Kolmogorov turbulence. Also, a generalization is proposed which allows intermittency effects to be modeled. © 1997 American Institute of Physics.

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I. INTRODUCTION

It is well known that a simple one-dimensional model of the Navier–Stokes system, namely the Burgers equation

\[
\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = \nu \frac{\partial^2 v}{\partial x^2},
\]

displays a dynamical behavior that is fundamentally different from the Navier–Stokes dynamics. First, Burgers equation possesses an infinite number of conserved quantities in the case of vanishing viscosity and contains only local interactions in x-space. Both properties are not shared by the Navier–Stokes equation. Second, the solutions of the Burgers equation do not demonstrate any spatial chaos at all, unless they are forced to do so by an intensive forcing acting at small scales. At high Reynolds number, solutions of Eq. (1) form strong shocks which dissipate most of the flow energy. The shock formation was studied for solutions decaying freely from some initial conditions or driven by a large-scale random forcing.\(^1\)\(^-\)\(^4\)

It was shown that shocks lead to statistical properties which are quite different from those of the classical Kolmogorov turbulence or real turbulence observed in experiments. In particular, the energy spectrum \(E(k)\) and the velocity structure functions \(S_n = \langle [v(x+r) - v(x)]^n \rangle\) scale, respectively, as \(E(k) \propto k^{-2}\) and \(S_n \propto r^n\)\(^1\)\(^-\)\(^4\).

Chekhlov and Yakhot\(^5\) studied the Burgers equation numerically, where the viscous term was replaced by the hyperdissipation term \(-\nu \partial^{12}v/\partial x^{12}\) and a random forcing has been added to the right hand side of Eq. (1). The white-in-time random forcing \(f(x,t)\) was defined by the correlation function

\[
\langle f(k,t)f(k',t') \rangle = 2(2\pi)^2 Dk^{-1} \delta(k+k') \delta(t-t'),
\]

where \(f(k,t)\) is the Fourier representation of \(f(x,t)\). In this case, the forcing at small scales was strong enough to introduce small-scale spatial randomness into the solution. The wavenumber dependence in (2) was chosen to provide an “almost-constant” energy flux \(\Pi(k) \propto \ln(kk_0)\) in the Fourier space. Thus the dynamics of the forced equation became more similar to that of Kolmogorov turbulence where \(\Pi(k) = \text{const}\) at \(\nu \rightarrow 0\).

Typical solutions calculated in Ref. 5 were superpositions of sawtooth structures characteristic of the Burgers equation and a small-scale random component of the velocity field. It was shown that the solution has the Kolmogorov energy spectrum \(E(k) \propto k^{-5/3}\). On the other hand, the scaling exponents \(a_{2n}\) of the structure functions, defined by \(S_n \propto r^{a_n}\), were found to be almost independent of \(n\), \((a_n \approx 0.91\) at \(n = 4, 6, 8\)). Clearly, this scaling is quite different from the Kolmogorov one \(a_n = n/3\) and indicates that the higher order correlation functions are still dominated by shocks. This difference is not surprising in view of the fact that the Burgers equation is local in space while the Navier–Stokes equation contains non-local interaction due to the pressure term. The importance of non-locality was highlighted by an exactly solvable model developed by Constantin et al.\(^6\)

In this paper, a simple generalization of the Burgers equation is proposed, taking into account spatially non-local interaction. Despite its simplicity, the model retains several of the most important structural features of Navier–Stokes turbulence. The solutions exhibit statistical properties which are surprisingly similar to classical Kolmogorov turbulence or to real 3D turbulence as observed in laboratory experiments and direct numerical simulations. Since this nonlocal Burgers equation is the next nontrivial step from Burgers to Navier–Stokes turbulence, it can serve as an ideal model to test concepts like the operator–product expansion\(^7\) and the instanton approach.\(^8\)\(^9\)

The modified equation is discussed in Sec. II. Also in Sec. II, the numerical technique is outlined. The results of calculations are given in Sec. III and Sec. IV. Concluding remarks are given in Sec. V and main conclusions are summarized in Fig. 12.
II. NON-LOCAL BURGERS EQUATION AND NUMERICAL METHOD

We consider the family of generalized Burgers equations

$$\frac{\partial v}{\partial t} + w \frac{\partial v}{\partial x} = \nu (-1)^p + 1 \frac{\partial^2 v}{\partial x^{2p}} + f,$$

(3)

with

$$w(x,t) = \alpha v(x,t) + (1 - \alpha) \mathcal{H} v(x,t)$$

(4)

indexed by a parameter $\alpha \in [0,1]$. The random forcing $f(x,t)$ is defined by (2) and

$$\mathcal{H} v(x,t) = \frac{1}{\pi} \int_{-\infty}^{\infty} v(x',t) \frac{dx'}{x-x'},$$

(5)

is the Hilbert transform of the velocity field.

Using the Fourier representation the transform can be defined as

$$\mathcal{H} v_k(t) = i \text{sgn}(k) v_k(t).$$

(6)

This function was used as a nonlinear term to describe the Marangoni convection in the limit of infinite Marangoni number.\textsuperscript{10}

Clearly, $\alpha = 1$ gives the ordinary Burgers equation. At $\alpha < 1$, the nonlinear interaction in (3) is non-local in the sense that the instantaneous properties of the solution at some point $x_0$ are defined by the velocity $v(x,t)$ in the whole range under consideration rather than by only the values of the velocity and its derivatives at $x = x_0$.

Equation (3) is solved at $x \in [0,2\pi]$ with periodic boundary conditions. A pseudo-spectral Fourier method is used. The nonlinear term is treated by FFT fully dealiased using the 3/2-rule.

Instead of the ordinary dissipation, the “hyperviscous” dissipation term $\nu (-1)^p + 1 \frac{\partial^2 v}{\partial x^{2p}}$ with $p = 6$ is applied in (3) to obtain a wider inertial range in the spectra. There are indications that the influence of hyperdissipation on the inertial range properties is not significant (see Ref. 5).

It must be noted that, as opposed to the ordinary Burgers equation, solution of Eq. (3) by pseudo-spectral method requires the inclusion of the Fourier mode $v_0$ with zero index. One can see that even if the mode $v_0$ is initially zero, the non-local nonlinear term at $\alpha < 1$ leads to positive energy transfer into this mode. At $k = 0$, we have the equation

$$\frac{\partial v_0}{\partial t} = \mathcal{N}_0 - 2(1 - \alpha) \sum_{k=1}^{N/2-1} k |v_k|^2 < 0,$$

with a solution $v_0 \to -\infty$. When considering the case $\alpha = 0$ the mode $v_0$ does not affect other modes since due to (6)

$$\mathcal{H} v_0(t) = i \text{sgn}(0) v_0(t) = 0.$$

However, at $0 < \alpha < 1$, infinitely decreasing $v_0$ influences other modes via the part $\alpha v \partial v / \partial x$ of the nonlinear term. It can be shown that at large times $t$ the solution becomes almost periodic dominated by $v_0$.

To banish the dominating influence of the zero mode we add damping at a largest scale and solve the equation

$$\frac{\partial v_0}{\partial t} = \mathcal{N}_0 - \lambda v_0$$

(7)

along with Eq. (3) for the other modes.

Clearly, the magnitude of $v_0$ can be controlled choosing the parameter $\lambda$. If $\lambda T > 1$, where $T$ is the typical time scale of variation of $\mathcal{N}_0$, we obtain, to leading order in $(\lambda T)^{-1}$ a slaved solution\textsuperscript{11}

$$v_0 \approx -\mathcal{N}_0 / \lambda,$$

that is reproduced easily by time-stepping technique used here. The value $\lambda = 10$ is taken for the calculations.

As in Ref. 5, the forcing is simulated in Fourier space as

$$f(k,t) = A_f / \sqrt{\Delta t} |k|^{-\frac{3}{2}} \sigma_k,$$

where $\Delta t$ is the time step and $\sigma_k$ is a Gaussian random function with $\langle |\sigma_k|^2 \rangle = 1$. The force cutoff $k_c$ such that $f(k,t) = 0$ at $k > k_c$ is taken to be inside the dissipation range of the energy spectrum.

The “slaved leap frog” scheme\textsuperscript{11} is employed for temporal discretization. To suppress the numerical oscillations inherent in leap-frog methods the solutions obtained in two consecutive time layers are averaged every 25 steps.

The spectral resolution applied here includes $N = 8192$ Fourier modes. The other parameters used are time step $\Delta t = 10^{-4}$, hyperviscosity coefficient $\nu = 9.345 \times 10^{-38}$, forcing amplitude $A_f = \sqrt{2} \times 10^{-3}$, and force cutoff $k_c = 3895/N - 2 = 1.2$.\textsuperscript{10}

III. PURE NON-LOCAL CASE $\alpha = 0$

Integration of Eq. (3) with $\alpha = 0$ was performed over the range $0 \leq t \leq 450$. A largest eddy turnover time can be estimated as $\tau_0 = \pi / v_{ms} \approx 150$. The statistically steady state was achieved after approximately $t = 60$.

Fig. 1 presents the time dependence of the total energy $E(t)$ of the system. Strong fluctuations of the energy can be seen.

A typical snapshot of the velocity field is shown in Fig. 2. One can see that any shocks characteristic of the ordinary Burgers equation disappear in the solution. Instead, the mean velocity profile has a zigzag form with several nearly singular peaks.
The time-averaged energy spectrum $E(k)$ presented in Fig. 3 is calculated as an average of instantaneous spectra $E(k,t_j)$, $t_j=t_s+jT$, where $t_s$ is the time of establishing the statistically steady solution and $T=2.5$. The scaling $E(k) \propto k^{-5/3}$ was observed for ordinary Burgers equation with the same forcing.\(^5\) Our calculations discussed below provide the scaling exponent of a second order structure function $a_2=0.6307$. Therefore, the power-law factor $k^{-1.6307}$ rather than $k^{-5/3}$ can be supposed for the energy spectrum. It can be seen in Fig. 3 that the calculated spectrum is well approximated by this law.

The velocity structure functions $S_n = \langle (v(x+r) - v(x))^n \rangle$ are evaluated for $n=2, \ldots, 9$ in the following manner. At each $t=t_j$, the functions $h_l(x)$ are calculated in physical space using FFT. Then, raising to the powers $n$ is performed and the functions $h^n_l(x)$ are transformed back to the spectral space. The space averages

$$C_n(t_j, r_s) = \int_0^{2\pi} h^n_l(x) dx$$

can then be obtained as zero index coefficients of the Fourier expansions. The structure functions $S_n(r_s)$ are calculated as time averages of $C_n(t_j, r_s)$.

The results are shown in Fig. 4 and Table I. One can see that structure functions can be well approximated by power laws $S_n \approx r^{a_n}$ at least in the range $\log(r) \in [-2.3; -0.4]$.

The scaling exponents are quite different from those obtained for ordinary Burgers equation with the same forcing.\(^5\) Moreover, they are very close to the exponents $a_n = n/3$ in self-similar Kolmogorov turbulence.

The discussion above allows a conclusion that the pure non-local form of the nonlinear interaction term removes anomalous scaling and intermittency characteristic of the Burgers equation. Instead, self-similar solutions appear, their statistical properties being strikingly close to those of the classical Kolmogorov turbulence.

Additional information on the statistical properties of the solution can be obtained if one considers probability distribution functions (pdfs) of the velocity $v$ and its derivative $s = dv/\partial x$. We cannot use time series of $v$ and $s$ at a given space location since our integration times are too short. Therefore, space distributions are used. We calculate $v$ and $s$ at all grid points in the physical space. Repeating the procedure every 500 time steps we obtain a total of $\approx 7.5 \times 10^7$ values.

The resulting probability densities normalized by variance are compared with a Gaussian distribution in Fig. 5 and Fig. 6. Velocity pdf in Fig. 5 is close to Gaussian only at probability levels higher than $10^{-3}$. Contrastingly, pdf of velocity gradient $s$ shown in Fig. 6 coincides with Gaussian one in whole range of reliable levels of probability.

The probability distributions seem to display discrepancy between the statistics of Eq. (3) and three-dimensional turbulence. It is known from experimental data (see Ref. 12 for a review) and numerical simulations (see, e.g. Ref. 13) that the pdf of velocity in real turbulence is close to Gaussian, while the distribution of velocity gradient is closer to an exponential law and decreases much more slowly than Gaussian.

**Table I.** Scaling exponents $a_n$ of the velocity structure functions in purely non-local case $a=0$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_n$</td>
<td>0.6307</td>
<td>0.9467</td>
<td>1.2629</td>
<td>1.5793</td>
<td>1.8955</td>
<td>2.2114</td>
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</tr>
<tr>
<td>$a_n/n$</td>
<td>0.3154</td>
<td>0.3156</td>
<td>0.3157</td>
<td>0.3158</td>
<td>0.3159</td>
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</table>

There is also a discrepancy between our distributions shown in Figs. 5 and 6 and corresponding pdfs for the ideal Kolmogorov turbulence. It was shown\textsuperscript{14} that, within classical Kolmogorov phenomenology, if distribution of velocity at large scales is Gaussian, the pdf of velocity gradients has tails following to an exponential law with an $s^{-1/3}$ prefactor.

The probability distribution of velocity gradient shown in Fig. 6 differs significantly from that for the ordinary Burgers equation. The latter is dominated by shock structures which correspond always to negative velocity differences. It is known\textsuperscript{2} that the pdf of velocity gradients for the Burgers equation is highly asymmetric. This is characterized by a slowly decreasing nearly exponential tail at negative $s$ and a sharp cutoff at positive $s$. A similar behavior was detected in our low resolution simulations of the Burgers equation with intensive small-scale forcing\textsuperscript{2}.

IV. INTERMEDIATE CASE $\alpha = 0.15$

It is the result of numerous experimental and numerical studies that classical Kolmogorov theory of self-similar turbulence cannot explain all the properties of real flows (for a review see, e.g., Refs. 12 and 15). While giving a good approximation to the energy spectrum, this theory fails to predict scaling exponents $a_n$ of high-order structure functions. Experimental measurements\textsuperscript{16} of structure functions revealed strong deviation from the theory with $a_n < n/3$ at $n > 3$. A common explanation of this anomalous scaling is that velocity field in real turbulent flows is spatially intermittent at small scales due to the presence of small coherent structures formed by sufficiently strong singularities. The flow dynamics can be thought of as a result of interaction between the coherent structures dissipating most of the flow energy and a background field described well by the classical Kolmogorov theory.

Similarly, we can consider the dynamics of a solution of Eq. (3) at $0 < \alpha < 1$ as a combination of dynamics defined by purely non-local nonlinear interaction (part $(1 - \alpha) \mathcal{H} \nu(x,t) \delta / \delta x$ of the nonlinear term) and that of the ordinary Burgers equation (part $\alpha \nu(x,t) \delta \nu / \delta x$). It has been shown above that a solution in pure non-local case $\alpha = 0$ has the same statistical properties as a self-similar velocity field of the Kolmogorov theory. On the other hand, a shock appearing in a solution of ordinary Burgers equation can be treated as a one-dimensional counterpart of a coherent structure. Therefore, the solution at an intermediate value $\alpha = 0.15$ discussed below can be viewed as a one-dimensional model of an intermittent turbulent velocity field.

Integration was performed at $0 \leq t \leq 450$ with the same constants $\nu$, $A_f$, and $k_c$, and numerical parameters $N$ and $\Delta t$ as in the case $\alpha = 0$.

The time dependence of the energy $E(t)$ is qualitatively similar to that shown in Fig. 1 and is not discussed here. A typical velocity profile (Fig. 7) also seems to be similar to its counterpart in the pure non-local case (cf. Fig. 2). There is no drastic differences in the scaling of the energy spectrum shown in Fig. 8 (cf. Fig. 3). The only distinction is that the
The scaling exponent of the second order structure function is now $a_2 \approx 0.5861$. Accordingly, the power law factor $k^{-1.5861}$ is supposed for the energy spectrum.

The inspection of structure functions (see Fig. 9 and Table II) reveals the main distinction between cases $\alpha = 0$ and $\alpha = 0.15$. The approximation by power laws $S_n \propto r^{a_n}$ was calculated at $\log_{10} r \in [-2.3; -1]$. The scaling exponents $a_n$ display a deviation from the Kolmogorov law $n/3$ that looks very similar to the anomalous scaling detected for the real turbulent flows. The exponents are close to $n/3$ only at $n \leq 3$. At higher $n$, we obtain $a_n < n/3$, the deviation increasing with $n$.

This anomalous scaling allows us to state that the solution of Eq. (3) at $\alpha = 0.15$ is intermittent at inertial range scales. We are not able to detect any coherent structures by direct visual inspection of the velocity profile. Also, the visual inspection of the high-pass filtered velocity fields does not reveal the intermittent behavior. In the following, we show that the form of pdf of velocity gradient confirms the statement above.

The normalized probability distributions of velocity $v$ and velocity gradient $s$ are shown in Fig. 10 and Fig. 11, respectively. The pdfs were calculated using the same technique as in the case $\alpha = 0$. The $x$-independent part $v_0$ of the velocity was not taken into account. Form of the velocity pdf at $\alpha = 0.15$ (see Fig. 10) looks very similar to the case $\alpha = 0$ (cf. Fig. 5). Contrastingly, pdf of velocity gradient in Fig. 11 differs drastically from that shown in Fig. 6. At $\alpha = 0.15$, probability distribution begins to take on the properties of pdf characteristic for the ordinary Burgers equation. The probability distribution at positive $s$ declines faster than Gaussian, while the tail at negative $s$ decreases much slower than Gaussian and looks exponential at low probability levels. Obviously, such distribution suggests the presence of large negative gradients in velocity profile which can be considered as a kind of coherent structures.

V. DISCUSSION

The statistics of velocity fluctuations governed by a generalized Burgers equation with random forcing has been studied by high-resolution numerical analysis. The equation differs from the ordinary Burgers one by the non-local form of nonlinear term. It has been shown that this change of the nonlinear term leads to solutions with fundamental statistical properties which are similar to those of three-dimensional turbulence. The energy spectrum is found to be close to $k^{-5/3}$. Such scaling was obtained earlier for the ordinary Burgers equation and, obviously, is due to the applied forcing.

More interesting is the change of behavior of the velocity structure functions. The scaling exponents $a_n$ calculated with different nonlinear terms are shown in Fig. 12. In the case of ordinary Burgers equation the solution is dominated by coherent shocks and the high-intermittent scaling with $a_n \approx 0.9$ at $n \geq 4$ was obtained. Contrastingly, when the purely non-local nonlinear term with $\alpha = 0$ is used, shocks do not appear and the solution seems to be quite self-similar, the scaling exponents $a_n$ being very close to the $n/3$-law of the classical Kolmogorov theory. If a combined nonlinear

TABLE II. Scaling exponents $a_n$ of the velocity structure functions in intermediate case $\alpha = 0.15$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>2</th>
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<th>4</th>
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<th>6</th>
<th>7</th>
<th>8</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$a_n$</td>
<td>0.5861</td>
<td>0.8683</td>
<td>1.1371</td>
<td>1.3884</td>
<td>1.6187</td>
<td>1.8265</td>
<td>2.0127</td>
<td>2.1803</td>
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<tr>
<td>$a_n/n$</td>
<td>0.2931</td>
<td>0.2894</td>
<td>0.2843</td>
<td>0.2777</td>
<td>0.2698</td>
<td>0.2609</td>
<td>0.2516</td>
<td>0.2423</td>
</tr>
</tbody>
</table>
term with $\alpha = 0.15$ is applied, the solution becomes to some degree intermittent and the behavior of scaling exponents is similar to that observed experimentally and numerically for real turbulence. The exponents are much less than $n/3$ at large $n$, the deviation increasing with $n$.

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