‘Power’ Notes (Review)

1) The complex conjugate of \( z = a + jb \) is \( z^* = a - jb \), and vice versa.

\[
\begin{align*}
z^* \equiv z &= (a + jb)(a - jb) = a^2 + b^2 = |a + jb|^2 \\
a &= \Re[z + z^*] = \frac{z + z^*}{2} \\
b &= \Im[z - z^*] = \frac{z - z^*}{2i} \\
(z_1z_2^*)^* &= ((z_1)(z_2^*))^* = z_1^*z_2
\end{align*}
\]

2) \[
\begin{align*}
V(t)I(t) &= \frac{[V_0e^{j\omega t} + V^*e^{-j\omega t}] [I_0e^{j\omega t} + I^*e^{-j\omega t}]}{4} \\
&= \frac{[VI^* + V^*I + V*I + V*I*]}{4}
\end{align*}
\]

The \(< V(t)I(t) >\), i.e. the average value of \( V(t)I(t) \) over one cycle, is

\[
< V(t)I(t) > = \frac{1}{2\pi\omega} \int_{-\infty}^{\infty} V(t)I(t) \, dt = \frac{[VI^* + V^*I]}{4} = \frac{[VI^* + (VI)^*]}{4}
\]

\[
= \frac{1}{2} \Re[VI^*]
\]

3) In general \( V \) and \( I \) are complex numbers; \( V = V_m \angle \theta, I = I_m \angle \phi \), and

\[
< V(t)I(t) > = \frac{1}{2} \Re[VI^*] = \frac{1}{2} V_m I_m \cos(\theta - \phi)
\]

For an inductance or a capacitance the magnitude of the angle \( \theta - \phi \) is 90˚, and the average power is zero (as expected, since a reactance only stores energy).

4) Analyze a sinusoidal circuit in the frequency domain as before; the formulas above enable calculation of average power without transforming back to the time domain.

5) A conclusion of some usefulness is derived next. The circuit diagram to the right describes a load impedance \( Z_L \), and the Thevenin equivalent of the remainder of the circuit 'seen' by \( Z_L \). A question often posed involves the efficient transfer of power from the source to the load. What load impedance should be used for a fixed Thevenin circuit to obtain maximum load power.

The average load power is

\[
\begin{align*}
\text{average load power} &= \frac{1}{2} \Re \left[ \left( \frac{V_T Z_L}{Z_T + Z_L} + \frac{V_T}{Z_T} \right)^* \right] \\
&= \frac{1}{2} \left| \frac{V_T}{Z_T + Z_L} \right|^2 \Re \left[ Z_L \right] \\
&= \frac{1}{2} \frac{|V_T|^2 R_L}{(R_L + R_T)^2 + (X_L + X_T)^2}
\end{align*}
\]
where \( Z_T = R_T + jX_T \), \( Z_L = R_L + jX_L \). This power is maximized if we choose \( X_L + X_T = 0 \); this minimizes the denominator. Note that the power is a positive quantity, equal to zero for \( R_L = 0 \), and approaching zero for \( R_L \to \infty \). Hence the power is a maximum for some intermediate value; differentiation determines this maximum to occur for \( R_L = R_T \).

To summarize: for maximum power transfer make \( Z_L = Z_T^* \); the maximum power is \( V_T^2/4R_L \).

6) **Effective** Value: this is a concept useful in comparing different periodic signals with waveforms, e.g., compare a triangular waveform to a sinusoidal waveform. The procedure is to compute the average power dissipated in a resistor, and derive from this a DC voltage or current value that produces the same power dissipation. Thus for, say, a current waveform with period \( T \) calculate

\[
P_{\text{ave}} = \frac{1}{T} \int_{t_0}^{t_0+T} I(t)^2 R \, dt
\]

and set this equal to \( I_{\text{eff}}^2 R \), i.e.,

\[
I_{\text{eff}} = \sqrt{\frac{1}{T} \int_{t_0}^{t_0+T} I^2(t) \, dt}
\]

The effective value also is known (from the mathematical operations involved) as 'root-mean-square' or 'rms' value.

The rms value of a sinusoidal waveform is the peak value/\( \sqrt{2} \). For a triangular waveform the rms value is \( I/\sqrt{3} \). Thus for the same peak value the sinusoid provides more power than the triangular waveform.

7) Problem 11.9 Irwin (4th ed.)

\[
e = \frac{e + 12}{2} + \frac{e}{j} + \frac{e}{4} \quad \text{and} \quad e = \frac{-8}{3+4j}
\]

\[
I_4 = \frac{e}{4} = \frac{-2}{3+4j}
\]

\[
I_2 = \frac{e + 12}{2} = \frac{14 + 24j}{3 + 4j}
\]

\[
\text{average Power} = \frac{1}{2} \text{Re} \left[ \left( \frac{14 + 24j}{3 + 4j} \right) \left( \frac{14 - 24j}{3 - 4j} \right) + \left( \frac{-2}{3 + 4j} \right) \left( \frac{-2}{3 - 4j} \right) \right]
\]

\[= 30.88 \quad + \quad 0.32
\]

\[= 31.2 \, \text{watts}
\]
8) Problem 11.12 Irwin (4th ed.)

\[-2Y_0 - \frac{24}{1} + \frac{2Y_0}{-j} + \frac{2Y_0}{2} = 0\]

\[V_0 = \frac{-24}{1-4j} = 5.82 \angle 104.04^\circ\]

\[I_5 = \frac{V_0}{1} = 5.82 \angle 104.04^\circ = -1.44 + j5.65\]

\[I_4 = \frac{2Y_0}{1} = 11.64 \angle 14.04^\circ = 11.29 + j2.82\]

\[I_3 = I_4 + I_5 = 9.88 + j3.47 = 13.01 \angle 40.6^\circ\]

\[I_2 = \frac{-2V_0}{-j} = 11.64 \angle 14.04^\circ = 11.29 + j2.82\]

\[I_1 = \frac{-2V_0-24}{1} = -21.18 + j11.3 = 24 \angle 152^\circ\]

Note: be sure to use proper quadrant.

\[P_2 = \frac{1}{2} \operatorname{Re}[I_5 I_5^* 2^*] = 33.87 \text{ watts}\]

\[P_{4V_0} = \frac{1}{2} \operatorname{Re}[-I_3(4V_0)^*] = -\frac{1}{2} \left(13.01 \times 4 \times (5.82 \cos(40.6^\circ) - 104.04^\circ)\right)\]

\[= -67.71 \text{ watts}\]

\[P_1 = \frac{1}{2} \operatorname{Re}[I_1 I_1^* 1] = 288 \text{ watts}\]

\[P_{24} = \frac{1}{2} \operatorname{Re}[24 I_1^*] = -254.3 \text{ watts}\]

\[P_1 + P_2 = \text{power consumed} = 288 + 33.87 = 321.87 \text{ watts}\]

\[P_{4V_0} + P_{24} = \text{power generated} = -67.71 - 254.3 = -322 \text{ watts}\]

9) Problem 11.20 Irwin (4th ed.)

\[v_T = 2(2 + \frac{-2j}{1-2j}) = 6 \frac{1}{1-2j} = -2(1+6j)\frac{1}{1-2j}\]

\[z_T = 2 + \frac{-2j}{1-2j} = 2\frac{7j}{5}\]

For maximum power transfer

\[Z_L = \frac{2(7j)}{5} = 2.8 + 0.4j\]

\[P = \frac{1}{2} \left| \frac{1}{2} \left( \frac{-2(1+6j)}{1-2j} \right) \right|^2 = 1.32 \text{ watts}\]